Shifting theorem –

(i) If $L\{F(t)\} = f(s)$ then $L\{F(t-a)\,u(t-a)\} = e^{-as} f(s)$

proof: $L\{F(t-a)\,u(t-a)\}$

$= \int_0^\infty e^{-st} F(t-a)\,u(t-a)\,dt$

$= \int_0^a e^{-st} F(t-a)\cdot(0)\,dt + \int_a^\infty e^{-st} F(t-a)\cdot 1\cdot dt$

$= 0 + \int_a^\infty e^{-st} F(t-a)\,dt$ Let $t-a = x$

t	a	∞
x	0	∞

$= \int_0^\infty e^{-s(a+x)} \cdot F(x)\,dx$

$= e^{-as} \int_0^\infty e^{-sx} F(x)\,dx$

$= e^{-as} f(s)$

$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$ unit step function

(ii) If $L\{F(t)\} = f(s)$ then $L\{F(t)\,u(t-a)\} = e^{-as} L\{F(t+a)\}$

proof: $L\{F(t)\,u(t-a)\} = \int_0^\infty e^{-st} F(t)\,u(t-a)\,dt$

$= \int_0^a e^{-st} F(t)\cdot(0)\,dt + \int_a^\infty e^{-st} F(t)\cdot 1\cdot dt$

$= 0 + \int_a^\infty e^{-st} F(t)\,dt = \int_0^\infty e^{-s(a+y)} F(y+a)\,dy$ Let $y = t-a$

t	a	∞
y	0	∞

$= e^{-as} \int_0^\infty e^{-sy} F(y+a)\,dy = e^{-as} \int_0^\infty e^{-st} F(t+a)\,dt$

$= e^{-as} L\{F(t+a)\}$

Co-ordinator IQAC Krishnath College Performance

Convolution theorem :

If $L\{F_1(t)\} = f_1(s)$ and $L\{F_2(t)\} = f_2(s)$ then

$$L\left\{\int_0^t F_1(x)\, f_2(t-x)\, dx\right\} = f_1(s) \cdot f_2(s)$$

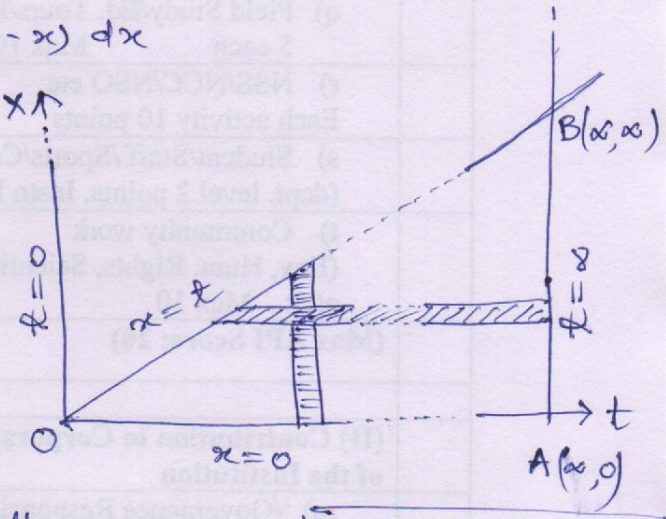
or

$$L^{-1}\{f_1(s)\, f_2(s)\} = \int_0^t f_1(x)\, F_2(t-x)\, dx$$

Proof: We have

$$L\left\{\int_0^t f_1(x) f_2(t-x)\, dx\right\} = \int_0^\infty e^{-st}\left\{\int_0^t f_1(x) F_2(t-x)\, dx\right\} dt$$

$$= \int_{t=0}^{t=\infty}\int_{x=0}^{x=t} e^{-st} f_1(x) F_2(t-x)\, dx\, dt$$



coordinates of B is (∞, ∞) and A is $(\infty, 0)$

The limits are $t=0$, $t=\infty$ and $x=0$, $x=t$.
So the region of integration is $\triangle OAB$.

Now changing the order of integration we get

$$L\left\{\int_0^t f_1(x)\, F_2(t-x)\, dx\right\}$$

$$= \int_{x=0}^{x=\infty}\int_{t=x}^{t=\infty} e^{-st} f_1(x)\, F_2(t-x)\, dt\, dx$$

$$= \int_0^\infty f_1(x)\, dx \int_x^\infty e^{-st} f_2(t-x)\, dt$$

put $t-x = y$
$dt = dy$

t	x	∞
y	0	∞

$$= \int_0^\infty f_1(x)\, dx \int_0^\infty e^{-s(x+y)} F_2(y)\, dy$$

$$= \int_0^\infty e^{-sx} f_1(x)\, dx \int_0^\infty e^{-sy} F_2(y)\, dy$$

$$= f_1(s)\, f_2(s)$$ Proved

Co-ordinator IQAC Krishnath College Berhampore

Inverse Laplace transform

Definition : If the Laplace transform of a function $F(t)$ is $f(s)$ i.e, $L\{F(t)\} = f(s)$ then $F(t)$ is called an inverse Laplace transform of $f(s)$.

We also write

$$F(t) = L^{-1}\{f(s)\}$$

L^{-1} is called the inverse Laplace transform operator.

Example : $L\{t^n\} = \frac{n!}{s^{n+1}}$

$$\therefore t^n = L^{-1}\left\{\frac{n!}{s^{n+1}}\right\}$$

Few important relations —

(1) $L^{-1}\left\{\frac{1}{s}\right\} = 1$

(2) $L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!}$

(3) $L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

(4) $L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh(at)$

(5) $L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\sinh(at)$

(6) $L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin(at)$

(7) $L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos(at)$

Few important relations –

(8) $L^{-1}\{f(s-a)\} = e^{at} f(t)$

(9) $L^{-1}\left\{\frac{1}{(s-a)^2+b^2}\right\} = \frac{1}{b} e^{at} \sin(bt)$

(10) $L^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = e^{at} \cos(bt)$

(11) $L^{-1}\left\{\frac{1}{(s-a)^2-b^2}\right\} = \frac{1}{b} \cdot e^{at} \sinh(bt)$

(12) $L^{-1}\left\{\frac{s-a}{(s-a)^2-b^2}\right\} = e^{at} \cosh(bt)$

(13) $L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = \frac{1}{2a^3} (\sin at - at \cos at)$

(14) $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{1}{2a} t \sin(at)$

(15) $L^{-1}\left\{\frac{s^2-a^2}{(s^2+a^2)^2}\right\} = t \cos at$

(16) $L^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\} = \frac{1}{2a} [\sin at + at \cos at]$

Exercise —

Find the inverse Laplace transform of

(i) $\frac{1}{s-2}$

(ii) $\frac{1}{s^2-9}$

(iii) $\frac{s}{s^2-16}$

(iv) $\frac{1}{s^2+25}$

(v) $\frac{s}{s^2+9}$

(vi) $\frac{1}{(s-2)^2+1}$

(vii) $\frac{s-1}{(s-1)^2+4}$

(viii) $\frac{1}{(s+3)^2-4}$

(ix) $\frac{s+2}{(s+2)^2-25}$

(x) $\frac{1}{2s-7}$

(xi) $\frac{s^2+s+2}{s^{3/2}}$

(xii) $\frac{s-2}{6s^2+20}$

Co-ordinator IQAC Krishnath College

[VI] Inverse Laplace transform of derivatives.

If $L^{-1}\{f(s)\} = F(t)$, then $L^{-1}\{f^n(s)\} = (-1)^n t^n F(t)$.

Proof:- We already know that (from the Laplace transform of derivatives) (page-7)

$$L\{t^n F(t)\} = (-1)^n f^n(s)$$

$$\therefore t^n F(t) = L^{-1}\{(-1)^n f^n(s)\}$$

$$= (-1)^n L^{-1}\{f^n(s)\}$$

or, $$L^{-1}\{f^n(s)\} = (-1)^n t^n F(t)$$ (Proved)

[VII] Division by s.

If $L^{-1}\{f(s)\} = F(t)$ then $L^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t F(u)\,du$

Proof: We already know that (from Laplace transform of integral, page-9)

$$\frac{1}{s} f(s) = L\left\{\int_0^t F(u)\,du\right\}$$

Therefore, $$L^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t F(u)\,du$$

Problem: Using the relation of inverse Laplace transform of derivatives find $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$.

Ans:- we have

$$L\{\sin at\} = \frac{a}{s^2+a^2}$$

$$\therefore L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a}$$

$$L^{-1}\{f^n(s)\} = (-1)^n t^n F(t)$$

or $$L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+a^2}\right)\right\} = (-1)^1 \cdot t^1 \cdot \frac{\sin(at)}{a} = (-1)\frac{t\sin(at)}{a}$$

or, $$L^{-1}\left\{-\frac{2s}{(s^2+a^2)^2}\right\} = (-1)\frac{t}{a}\sin(at)$$

or, $$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{1}{2a} t \sin(at)$$ Ans.

Problem: Using the relation of inverse Laplace transform of integral find $L^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\}$

Ans: $L\{e^{at}\} = \frac{1}{s-a}$ $\therefore L\{e^{-at}\} = \frac{1}{s+a} \Rightarrow L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$

$$\therefore L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$$

or $$L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s+1}\right)\right\} = (-1)^1 \cdot t^1 \cdot e^{-t} = -te^{-t}$$

or $$L^{-1}\left\{\frac{1}{(s+1)^2}\right\} = te^{-t}$$

$$\therefore L^{-1}\left\{\frac{1}{s(s+1)^2}\right\} = \int_0^t x e^{-x}\,dx = \left[x\cdot\frac{e^{-x}}{(-1)} - 1\cdot\frac{e^{-x}}{(-1)^2}\right]_0^t$$

$$= (-te^{-t} - e^{-t} + 1) = 1 - e^{-t}(1+t)$$

and $$L^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\} = \int_0^t \left[1 - e^{-x}(1+x)\right]dx$$

$$= t - \left[(1+x)\frac{e^{-x}}{(-1)} - 1\cdot\frac{e^{-x}}{(-1)^2}\right]_0^t$$

$$= t - \left[-(1+t)e^{-t} - e^{-t} + 1 + 1\right]$$

$$= te^{-t} + 2e^{-t} + t - 2$$ Ans.

Co-ordinator IQAC Krishnanath College Berhampore

Problem on convolution theorem

(1) Find $L^{-1}\left\{\frac{1}{(s+a)(s+b)}\right\}$

> If $L\{F_1(t)\} = f_1(s)$ & $L\{F_2(t)\} = f_2(s)$ then
> $f_1(s) f_2(s) = L\left\{\int_0^t F_1(x) \cdot f_2(t-x)\, dx\right\}$

Ans: Let $f_1(s) = \frac{1}{(s+a)}$ and $f_2(s) = \frac{1}{s+b}$

Then $f_1(t) = e^{-at}$ and $F_2(t) = e^{-bt}$

$$\therefore L^{-1}\{f_1(s) f_2(s)\} = \int_0^t F_1(x)\, f_2(t-x)\, dx$$

$$= \int_0^t e^{-ax} \cdot e^{-b(t-x)}\, dx$$

$$= e^{-bt} \int_0^t e^{-ax} \cdot e^{bx}\, dx$$

$$= e^{-bt} \int_0^t e^{(b-a)x}\, dx$$

$$= e^{-bt} \left| \frac{e^{(b-a)x}}{(b-a)} \right|_0^t = \frac{e^{-bt}}{b-a}\left\{e^{(b-a)t} - 1\right\}$$

$$= \frac{e^{-at} - e^{-bt}}{b-a}$$ **Ans.**

Another rule –

$$\frac{1}{(s+a)(s+b)} = \frac{1}{b-a}\left[\frac{1}{s+a} - \frac{1}{s+b}\right]$$

$$\therefore L^{-1}\left\{\frac{1}{(s+a)(s+b)}\right\} = \left(\frac{1}{b-a}\right) L^{-1}\left\{\frac{1}{s+a}\right\} - \frac{1}{(b-a)} L^{-1}\left\{\frac{1}{s+b}\right\}$$

$$= \frac{1}{(b-a)} e^{-at} - \frac{1}{(b-a)} e^{-bt}$$

$$= \frac{(e^{-at} - e^{-bt})}{(b-a)}$$ **Ans.**