B.Sc. 2nd semester-2020
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Paper-HCC-III (Unit-2)

Theorem 1. *(Bolzano-Weierstrass Theorem for set): Every bounded, infinite subset of real numbers* $(\mathbb{R})$ *has at least one limit point in* $\mathbb{R}$.

We shall prove this theorem in different ways.

Proof. Let $S \subseteq \mathbb{R}$ be a bounded and infinite set. Since, S is a bounded set.
Hence, $\exists M > 0$ $S \subseteq [-M, M]$. Now, we define a set $T \subseteq \mathbb{R}$ as follows:

$$T = \{x : \text{infinitely many points of } S \text{ are} \geq x\}.$$

(Let us consider $S = [1, 2020]$ and find the set T to understand the definition of T.)
Then, $M \neq \phi$, since $-M \in S$. Because all the points of S are $\geq -M$ and S has infinite number of points.
Again, S is bounded above by M. Since, if $x > M$, then no point of S is $\geq x$.
Therefore, supremum of T exists, say $\sup T = \xi \in \mathbb{R}$. Claim: ξ is a limit point of S.
Therefore, we need to show that every Nbd of ξ contains infinitely many points of S. Let us consider any $\delta > 0$.
Since, $\sup T = \xi$, hence $\exists t \in T$ such that $t > \xi - \delta$.
Again, since $t \in T$, hence, there are infinitely many points of S that are $\geq t$. On the other hand, $\xi + \delta \notin T$(*why?*).
Thus, there are only finitely many points of S that are $\geq \xi + \delta$. This implies, there are infinitely many points of S that are in $[t, \xi + \delta) \subseteq (\xi - \delta, \xi + \delta)$. Hence, $\xi \in \mathbb{R}$ is a limit point of S. □

The next prove is easier than above prove. However, we need to use the nested interval theorem to prove the Bolzano-Weierstrass theorem for set.

Theorem 2. *(Nested interval theorem) Let* $\{I_n\}$ *be a nested sequence of closed, bounded intervals. That is* $I_1 \supseteq I_2 \supseteq \cdots$. *Then*

$$\bigcap_{n \in \mathbb{N}} I_n \neq \phi$$

Moreover, if $|I_n| \to 0$ *as* $n \to \infty$, *then*

$$\bigcap_{n \in \mathbb{N}} I_n \text{ is a singleton set.}$$

∘ Second proof of Bolzano-Weierstrass theorem for set.

Proof. Let $S \subseteq \mathbb{R}$ be a bounded and infinite set. Since, S is a bounded set. Hence, $\exists M > 0$ $S \subseteq [-M, M]$.
Therefore, $[-M, M]$ contains infinitely many points of S.
∘ We now divide $[M, M]$ into $[-M, 0]$ and $[0, M]$. At least one of $[-M, 0]$ or $[0, M]$ has been contained infinitely many points of S.
∘ Without any loss of generality, let us consider $[0, M]$ contains infinitely many points of S. and denoted by $S_1 = [0, M]$
∘ Notice that the length of $S_1(|S_1|)$ is $M = \frac{2M}{2}$.
∘ Again divide S_1 into two closed intervals of equal length. At least one, say, S_2, contains infinitely many points of S and the length of S_2 is $\frac{2M}{2^2} = \frac{M}{2}$.
∘ In this similar way we can construct $S_1, S_2, \cdot\cdot$, so on. Therefore, for each $n \in \mathbb{N}$ S_n is a closed interval of length $\frac{2M}{2^n} = \frac{M}{2^{n-1}}$ containing infinitely many points of S.
∘ Moreover, $S_1 \supseteq S_2 \supseteq \cdots S_n \supseteq S_{n+1} \cdots$. and length of S_n is $|S_n| = \frac{M}{2^{n-1}} \to 0$ as $n \to \infty$. By using nested interval theorem

$$\bigcap_{n \in \mathbb{N}} S_n = \{\xi\}.$$

∘ Claim: ξ is a limit point of S.
For any $\delta > 0$, consider $n \in \mathbb{N}$ in such a way that $\frac{2M}{2^n} = \frac{M}{2^{n-1}} < \delta$. Therefore,

$$(\xi - \delta, \xi + \delta) \supset [\xi - \frac{M}{2^{n-1}}, \xi + \frac{M}{2^{n-1}}] \supset S_n.$$

Since, S_n contains infinitely many points of S, hence $(\xi - \delta, \xi + \delta)$ contains infinitely many points of S. Therefore, ξ is a limit point of S. □

Last one is a short proof of the Bolzano-Weierstrass theorem. For that purpose, the following result is to be needed.

Theorem 3. *Let S be a set, and let $\leq$ be a linear order on S. If for every nonempty $X \subseteq S$, both $\inf(X)$ and $\sup(X)$ exist and belong to X, then S is finite.*

∘ Third proof of Bolzano-Weierstrass theorem for set.

Proof. Let S be an infinite, bounded subset of $\mathbb{R}$, and assume S has no limit point. Suppose $X \subseteq S$ is nonempty. Then $\inf(X) \in X$, and $\sup X \in X$. Theorem 3 implies that S is finite. This is a contradiction. Thus, our assumption is not true. Hence, every infinite, bounded subset of $\mathbb{R}$, has at least one limit point in $\mathbb{R}$. □

Note. *Do not write third proof in your university exam.*

Question 1. *Verify the Bolzano-Weierstrass theorem for the the* $\left\{1 + \frac{2020}{n} : n \in \mathbb{N}\right\}$.

Question 2. *Is converse of the Bolzano-Weierstrass theorem true? Justify the answer.*

Question 3. *Can we apply the Bolzano-Weierstrass theorem for the set of integer? Justify the answer.*