Laplace transform of derivatives :

If $L\{f(t)\} = f(s)$, then

$$L\{F^n(t)\} = s^n f(s) - s^{n-1} f(0) - s^{n-2} F'(0) - \cdots$$

$$\cdots - s F^{n-2}(0) - F^{(n-1)}(0)$$

where $F^n(t) = \frac{d^n F(t)}{dt^n}$

Proof: Let $L\{f(t)\} = f(s)$, then

$$L\{F'(t)\} = \int_0^\infty e^{-st} \frac{df(t)}{dt}\, dt$$

$$= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty e^{-st} (-s) f(t)\, dt$$

$$= -F(0) + s f(s)$$

$$\therefore L\{f'(t)\} = s f(s) - F(0) \longrightarrow (1).$$

In view of this,

$$L\{G'(t)\} = s g(s) - G(0) = s L\{G(t)\} - G(0)$$

Taking $G(t) = f'(t)$, we get

$$L\{F''(t)\} = s L\{f'(t)\} - f'(0)$$

$$= s[s f(s) - F(0)] - f'(0)$$

$$= s^2 f(s) - s F(0) - F'(0) \longrightarrow (2).$$

Generalized the result (1) and (2), we get the required result.

Corollary: To prove that $L[F'''(t)] = s^3 f(s) - s^2 F(0) - s F'(0) - F''(0)$

(4) prove that $L[f'''(t)] = s^3 f(s) - s^2 F(0) - sF'(0) - F''(0)$

Ans:

$$L\{f'(t)\} = sf(s) - F(0) \longrightarrow (1)$$

$$L\{F''(t)\} = s^2 f(s) - sf(0) - F'(0) \longrightarrow (2).$$

By equation (2),

$$L\{H''\} = s^2 h(s) - s\,H(0) - H'(0)$$

or $$L\{H''\} = sL\,L\{H\} - sH(0) - H'(0)$$

put $H = F'$

$$L\{F'''\} = s^2 L\{F'\} - sF'(0) - F''(0)$$

$$= s^2\left(sf(s) - F(0)\right) - sF'(0) - F''(0)$$

$$= s^3 f(s) - s^2 F(0) - sF'(0) - F''(0)$$

Co-ordinator IQAC Krishnath College Berhampore

Laplace Transform of Integral

If $L\{F(t)\} = f(s)$, then $\frac{1}{s} f(s) = L\left\{\int_0^t F(u)\,du\right\}$

Proof: Let $L\{F(t)\} = f(s)$ and $G(t) = \int_0^t F(u)\,du$. ⟶ (1)

We have to prove that $L\left\{\int_0^t F(u)\,du\right\} = \frac{1}{s} f(s)$

From (1) it is clear that

$$G(0) = \int_0^0 F(u)\,du = 0$$

$$G'(t) = \frac{d}{dt}\left(\int_0^t f(u)\,du\right) = F(t) \longrightarrow (2)$$

From previous relation $L\{F'(t)\} = sf(s) - F(0) = sL\{F(t)\} - F(0)$

$L\{F''(t)\} = s^2 f(s) - sF(0) - F'(0)$

$L\{F'''(t)\} = s^3 f(s) - s^2 F(0) - sF'(0) - F''(0)$

from previous relation

$$L\{G'(t)\} = sL\{G(t)\} - G(0)$$

i.e, $L\{F(t)\} = sL\{G(t)\} - 0 = sL\{G(t)\}$

using (2)

or, $\frac{1}{s} L\{F(t)\} = L\{G(t)\}$

or $\boxed{\frac{1}{s} f(s) = L\left\{\int_0^t F(u)\,du\right\}}$

If $\phi(p)$ be the Laplace transform of $f(x)$ and $\phi^r(p)$ be the Laplace transform of $\frac{d^r f}{dx^r}$, show that under certain condition $\phi^r(p) = p^r \phi(p)$.

Ans:- Let $L\{f(x)\} = \phi(p)$ and $L\left\{\frac{d^r f}{dx^r}\right\} = \phi^r(p)$

under certain condition we have to prove that

$$\phi^r(p) = p^r \phi(p)$$

$$L\left\{\frac{d^r f}{dx^r}\right\} = p^r L\{f(x)\}$$

By the theorem of Laplace transform of derivatives —

$$\phi^r(p) = L\left\{\frac{d^r f}{dx^r}\right\} = [\quad] L\{f^r\}$$

$$= p^r \phi(p) - p^{r-1} f(0) - p^{r-2} f'(0) - p^{r-3} f''(0) -$$

$$\cdots - p f^{r-2}(0) - f^{r-1}(0)$$

$= p^r \phi(p)$ if $f(0) = 0$, $f^n(0) = 0$ for $n = 1, 2, \cdots (r-1)$

$\therefore \phi^r(p) = p^r \phi(p)$

if (i) $f(x) = 0$ for $x = 0$

(ii) $f^n(x) = 0$ at $x = 0$ for $n = 1, 2, \cdots, (r-1)$

These are the required conditions.

Co-ordinator IQAC Krishnaveni College

Q If $L\{F(t)\} = f(s)$, then $L\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$ for $n = 1, 2, 3, \ldots$

Ans:- Let $L\{F(t)\} = f(s)$ and $\frac{d^n f(s)}{dx^n} = f^{(n)}(s)$

$$f(s) = \int_0^\infty e^{-st} F(t)\, dt$$

$$\frac{df(s)}{ds} = \frac{d}{ds} \int_0^\infty e^{-st} F(t)\, dt$$

$$= \int_0^\infty \frac{\partial}{\partial s} \{e^{-st} F(t)\}\, dt$$

$$= \int_0^\infty (-t)\, e^{-st} F(t)\, dt$$

or, $$(-1)\frac{df(s)}{ds} = \int_0^\infty e^{-st} (tF(t))\, dt = L\{tF(t)\}$$

or $$(-1) f'(s) = L\{tF(t)\}$$

This proves that the theorem is true for $n = 1$

Let us suppose that the theorem is true for $n = r$ so that

$$(-1)^r f^r(s) = L\{t^r F(t)\}$$

$$= \int_0^\infty e^{-st} t^r F(t)\, dt$$

$$(-1)^r \frac{d^{r+1} f(s)}{ds^{r+1}} = \int_0^\infty \frac{\partial}{\partial s} \left(e^{-st} t^r F(t)\right) dt$$

$$= (-1) \int_0^\infty t\, e^{-st} t^r F(t)\, dt$$

or $$(-1)^{r+1} f^{r+1}(s) = \int_0^\infty e^{-st} \left(t^{r+1} F(t)\right) dt$$

$$= L\{t^{r+1} F(t)\}$$

This proves that the theorem is true for $n = r+1$ if it is true for $n = r$. But we have already shown that the theorem is true for $n = 1$ and hence for $n = 1+1 = 2$ and therefore for $n = 2+1 = 3$ and so on. Hence it is true for any positive integer n.

Co-ordinator
IQAC
Krishnath College
Berhampore

If the Laplace transform of $f(t)$ is $\bar{f}(s)$, prove that the Laplace transform of $tf(t)$ is $-\frac{d\bar{f}}{ds}$.

Hence show that

$$\frac{1}{s^{1/2}} = L\left\{\frac{1}{\sqrt{(\pi t)}}\right\}$$

Ans: (i) $\bar{f}(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

Then $\frac{d\bar{f}}{ds} = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty \frac{\partial}{\partial s}\left(e^{-st} f(t)\right) dt$

$$\int_0^\infty (t) e^{-st} f(t)\, dt = \int_0^\infty (-t)\, e^{-st} f(t)\, dt$$

$$= -\int_0^\infty e^{-st}\left(t f(t)\right) dt$$

$$= -L\{t f(t)\}$$

$$\therefore \boxed{-\frac{d\bar{f}}{ds} = L\{t f(t)\}}$$

(ii) $L\left\{\frac{1}{\sqrt{\pi t}}\right\} = \int_0^\infty e^{-st} \frac{1}{\sqrt{\pi t}}\, dt = \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-st} t^{-1/2}\, dt$

$$= \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-x} \left(\frac{x}{s}\right)^{-1/2} \frac{dx}{s} \qquad \left| \begin{array}{l} st = x \\ dt = \frac{1}{s}dx \end{array} \right.$$

$$= \frac{1}{\sqrt{\pi s}}\int_0^\infty e^{-x}\, x^{\frac{1}{2}-1}\, dx$$

$$= \frac{1}{\sqrt{\pi s}} \cdot \Gamma(1/2) = \frac{1}{\sqrt{\pi s}} \cdot \sqrt{\pi}$$

$$= \frac{1}{s^{1/2}}$$

Co-ordinator IQAC Krishnnath College Berhampore

VII Suppose $f(s) = \int_0^\infty e^{-st} f(t)\,dt$ then show that

$$f'(s) = \int_0^\infty -t\, e^{-st} f(t)\,dt.$$

Ans. $$f'(s) = \frac{df(s)}{ds} = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\infty -t\, e^{-st} f(t)\,dt \quad \text{(proved)}$$

Theory : If $L\{F(t)\} = \{f(s)\}$, then $L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(x)\,dx$, provided the integral exists.

Ans: Let $L\{F(t)\} = f(s)$. Then

$$f(s) = \int_0^\infty e^{-st} F(t)\,dt$$

Integrating this w.r.t s from $s = s$ to $s = \infty$,

$$\int_s^\infty f(s)\,ds = \int_s^\infty \left\{\int_0^\infty e^{-st} f(t)\,dt\right\} ds$$

Since s and t are independent variables and hence order of integration in the repeated integral can be interchanged.

$$\therefore \int_s^\infty f(s)\,ds = \int_0^\infty f(t)\,dt \int_s^\infty e^{-st}\,ds = \int_0^\infty F(t)\,dt \left(\frac{e^{-st}}{-t}\right)_{s=s}^{\infty}$$

$$= \int_0^\infty \frac{1}{t} f(t)\, e^{-st}\,dt = \int_0^\infty e^{-st}\left\{\frac{f(t)}{t}\right\} dt$$

$$= L\left\{\frac{f(t)}{t}\right\}$$