Periodic function (Laplace transform of periodic function)

Theorem: Let a function $f(t)$ be periodic with period ω so that $F(t+n\omega) = F(t)$ for $n = 1, 2, 3, \ldots$

Then $L\{f(t)\} = \int_0^{\omega} \frac{e^{-st} f(t)}{1 - e^{-s\omega}}\, dt$

Proof: $L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt$

$t + n\omega = q$

t	∞	0
q	∞	$n\omega$

$$= \sum_{n=0}^{\infty} \int_{n\omega}^{(n+1)\omega} e^{-st} f(t)\, dt$$

$$\int_0^{\infty} e^{-st} f(t)\, dt = \int_0^{\omega} e^{-st} f(t)\, dt + \int_{\omega}^{2\omega} e^{-st} f(t)\, dt + \int_{2\omega}^{3\omega} e^{-st} f(t)\, dt + \cdots$$

$$= \sum_{n=0}^{\infty} \int_{n\omega}^{(n+1)\omega} e^{-st} f(t)\, dt$$

$$\therefore L\{f(t)\} = \sum_{n=0}^{\infty} \int_0^{\omega} e^{-s(x+n\omega)} F(x+n\omega)\, dx, \quad \text{where } t = x + n\omega$$

$$= \sum_{n=0}^{\infty} \int_0^{\omega} e^{-sx} e^{-ns\omega} f(x)\, dx$$

$$= \sum_{n=0}^{\infty} e^{-ns\omega} \int_0^{\omega} e^{-sx} f(x)\, dx$$

$$= \left(1 + e^{-s\omega} + e^{-2s\omega} + \cdots\right) \int_0^{\omega} e^{-sx} f(x)\, dx$$

$$= \frac{1}{1 - e^{-s\omega}} \int_0^{\omega} e^{-sx} f(x)\, dx \qquad \left[\because e^{-s\omega} < 1\right]$$

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(i) Initial value theorem : To prove that

$$\lim_{t \to 0} F(t) = \lim_{s \to \infty} sf(s)$$

(ii) final value theorem :

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sf(s)$$

Ans : (i) we ~~Knew~~ that

$$L\{F'(t)\} = sf(s) - F(0)$$

or $$\int_0^\infty e^{-st} f'(t)\, dt = sf(s) - F(0)$$

Making $s \to \infty$,

$$\lim_{s \to \infty} \int_0^\infty e^{-st} f'(t)\, dt = \lim_{s \to \infty} sf(s) - F(0)$$

or $$\lim_{s \to \infty} sf(s) = F(0) + \lim_{s \to \infty} \int_0^\infty e^{-st} f'(t)\, dt$$

$$= F(0) + \int_0^\infty \left(\lim_{s \to \infty} e^{-st}\right) f'(t)\, dt$$

$$= F(0) + \int_0^\infty 0 \cdot f'(t)\, dt = F(0)$$

$$= \lim_{t \to 0} F(t)$$

or, $$\boxed{\lim_{s \to \infty} sf(s) = \lim_{t \to 0} F(t)}$$ proved.

(ii) We Know that

$$L\{f'(t)\} = sf(s) - f(0)$$

or $$\int_0^\infty e^{-st} f'(t)\,dt = sf(s) - f(0)$$

Making $s \to 0$, we get

$$\lim_{s\to 0} sf(s) - f(0) = \lim_{s\to 0} \int_0^\infty e^{-st} f'(t)\,dt$$

$$= \int_0^\infty \left(\lim_{s\to 0} e^{-st}\right) f'(t)\,dt$$

$$= \int_0^\infty f'(t)\,dt$$

$$= \int_0^\infty \frac{df(t)}{dt}\,dt = f(t)\Big|_{t=0}^{t=\infty}$$

$$= \lim_{t\to\infty} f(t) - f(0)$$

$$\therefore \boxed{\lim_{t\to\infty} f(t) = \lim_{s\to 0} sf(s)}$$ Proved.

Theorem: Laplace transform of Laplace transform:

$$L\{L\{F(t)\}\} = L\left\{\int_0^\infty e^{-st} F(t)\,dt\right\} = \int_0^\infty e^{-us}\left(\int_0^\infty e^{-st} F(t)\,dt\right) ds$$

The area of integration being the whole positive quadrant. Now changing the order of integration we get

$$L\{L\{f(t)\}\} = \int_0^\infty F(t)\left(\int_0^\infty e^{-s(t+u)}\,ds\right) dt$$

$$= \int_0^\infty F(t)\left[\frac{e^{-s(t+u)}}{-(t+u)}\right]_{s=0}^{s=\infty} dt$$

$$= \int_0^\infty \frac{F(t)}{t+u}\,dt.$$

Problem : If $f(t) = t^2$, $0 < t < 2$, and $f(t+2) = f(t)$ find $L\{f(t)\}$.

Ans : Here, $f(t)$ is a periodic function with period $= 2$

We know that Laplace transform of periodic function $F(t)$ with period w is

$$L\{f(t)\} = \frac{1}{1-e^{-sw}} \int_0^w e^{-sx} F(x)\, dx$$

Therefore, $$L\{f(t)\} = L\{t^2\} = \frac{1}{1-e^{-2s}} \int_0^2 e^{-sx} x^2\, dx$$

Now $$\int_0^2 e^{-sx} x^2\, dx = \left| x^2 \frac{e^{-sx}}{-s} - (2x)\cdot\frac{e^{-sx}}{(-s)(-s)} + 2\cdot\frac{e^{-sx}}{(-s)^3} \right|_{x=0}^{x=2}$$

$$= \left[e^{-2s}\left(\frac{4}{-s} - \frac{4}{s^2} - \frac{2}{s^3}\right) + 0 + 0 + \frac{2e^0}{s^3} \right]$$

$$= \frac{2}{s^3}\left[1 - \left(\frac{2s^2}{2} + \frac{2s}{2} + \frac{1}{2}\right)e^{-2s}\right]$$

$$= \frac{2}{s^3}\left[1 - e^{-2s}\left(1 + 2s + 2s^2\right)\right]$$

$$\therefore L\{f(t)\} = \frac{2}{s^3}\left[1 - e^{-2s}\left(1+2s+2s^2\right)\right]\cdot\frac{1}{1-e^{-2s}}$$

$$= \frac{2\left[1 - e^{-2s}\left(1+2s+2s^2\right)\right]}{\left(1-e^{-2s}\right)s^3}$$ **Ans.**

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☒ find the Laplace transform of $\sinh(at)$, $\cosh(at)$

Ans: we know that $L\{e^{at}\} = \frac{1}{s-a}$

$$L\{\sinh at\} = L\left\{\frac{e^{at}-e^{-at}}{2}\right\} = \frac{1}{2}L\{e^{at}\} - \frac{1}{2}L\{e^{-at}\}$$

$$= \frac{1}{2}\cdot\frac{1}{s-a} - \frac{1}{2}\cdot\frac{1}{s+a} = \frac{a}{s^2-a^2} \quad (\text{if } s>|a|)$$

$$L\{\cosh at\} = L\left\{\frac{e^{at}+e^{-at}}{2}\right\} = \frac{1}{2}\left[L\{e^{at}\} + L\{e^{-at}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{s}{s^2-a^2} \quad (\text{if } s>|a|)$$

☒ find the Laplace transform of the function $f(t) = \frac{e^{at}-1}{a}$

Ans: $L\{f(t)\} = L\left\{\frac{e^{at}-1}{a}\right\} = \frac{1}{a}L\{e^{at}\} - \frac{1}{a}L\{1\}$

$$= \frac{1}{a}\cdot\frac{1}{s-a} - \frac{1}{a}\cdot\frac{1}{s} = \frac{1}{a}\left(\frac{s-s+a}{s(s-a)}\right)$$

$$= \frac{1}{s(s-a)}$$

☒ Evaluate $L\{f(t)\}$, if $F(t) = \begin{cases} (t-1)^2, & t>1 \\ 0, & 0<t<1 \end{cases}$

Soln: $L\{F(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 0\cdot e^{-st}\,dt + \int_1^\infty (t-1)^2 e^{-st}\,dt$

$$= 0 + \left[(t-1)^2\cdot\frac{e^{-st}}{(-s)} - 2(t-1)\cdot\frac{e^{-st}}{s^2} + 2\cdot\frac{e^{-st}}{-s^3}\right]_{t=1}^{t=\infty}$$

$$= \frac{2}{s^3}e^{-s} \quad \text{Ans.}$$

Prob: Find $L\{e^{-t}(3\sinh 2t - 5\cosh 2t)\}$

Soln: We know that if $L\{f(t)\} = f(s)$ then $L\{e^{at}F(t)\} = f(s-a)$

Also, $L\{\sinh 2t\} = \frac{2}{s^2-2^2}$

$L\{\cosh 2t\} = \frac{s}{s^2-2^2}$

$\therefore L\{e^{-t}(3\sinh 2t - 5\cosh 2t)\}$

$= 3.\frac{2}{(s+1)^2-2^2} - 5.\frac{(s+1)}{(s+1)^2-2^2}$

$= \frac{1-5s}{(s+1)^2-2^2}$

$= \frac{1-5s}{s^2+2s-3}$

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Problem: Find the Laplace transform of $\frac{\sin at}{t}$. Does the transform of $\frac{\cos at}{t}$ exist?

Ans: $L\{e^{iat}\} = \int_0^\infty e^{-st} e^{iat}\, dt = \int_0^\infty e^{-(s-ia)t}\, dt$

$= -\frac{1}{(s-ia)} \left(e^{-(s-ia)t} \right)_{t=0}^{t=\infty} = \frac{1}{s-ia}$

$\therefore L\{\cos at + i \sin at\} = \frac{(s+ia)}{(s-ia)(s+ia)} = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}$

$\Rightarrow L\{\cos at\} = \frac{s}{s^2+a^2}$

and $L\{\sin at\} = \frac{a}{s^2+a^2}$

Theorem – If $L\{f(t)\} = f(s)$

Then

$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty f(x)\, dx$

from previous relation we have

$L\left\{\frac{\sin at}{t}\right\} = \int_s^\infty f(x)\, dx = \int_s^\infty f(s)\, ds$

$= \int_s^\infty L\{f(t)\}\, ds$

$= \int_s^\infty \frac{a}{s^2+a^2}\, ds$

$= \left(\tan^{-1} s/a\right)_s^\infty = \tan^{-1}(\infty) - \tan^{-1}(s/a)$

$= \pi/2 - \tan^{-1}(s/a)$

$\frac{\frac{1}{a}}{1+\frac{x^2}{a^2}} = \frac{\frac{1}{a}}{\frac{a^2+x^2}{a^2}} = \frac{a}{x^2+a^2}$

Suppose $\tan^{-1}(s/a) = \phi$

$\tan\phi = s/a$

$\cot\phi = \tan\left(\frac{\pi}{2} - \phi\right)$

$L\left\{\frac{\cos at}{t}\right\} = \int_s^\infty L\{\cos at\}\, ds = \int_s^\infty \frac{s}{s^2+a^2}\, ds = \frac{1}{2}\left[\log(x^2+a^2)\right]_s^\infty$

$= \frac{1}{2}\left[\lim_{s\to\infty} \log(s^2+a^2) - \log(s^2+a^2)\right]$, which does not exist.

$\left(\because \lim_{s\to\infty} \log(s^2+a^2) \text{ is infinite}\right)$

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☑ prove that $\int_0^\infty \frac{e^{-t}\sin t}{t}\,dt = \pi/4$

Ans: As earlier we proved that $L\left\{\frac{\sin at}{t}\right\} = \pi/2 - \tan^{-1}(s/a)$

i.e, $\int_0^\infty e^{-st}\frac{\sin at}{t}\,dt = \pi/2 - \tan^{-1}(s/a)$

Taking $a=1$, $s=1$ we get

$$\int_0^\infty e^{-t}\frac{\sin t}{t}\,dt = \frac{\pi}{2} - \tan^{-1}(1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

☑ prove that $\int_0^\infty \frac{\sin t}{t}\,dt = \pi/2$

Ans: As earlier we proved that $L\left\{\frac{\sin at}{t}\right\} = \pi/2 - \tan^{-1}(s/a)$

i.e, $\int_0^\infty e^{-st}\frac{\sin at}{t}\,dt = \pi/2 - \tan^{-1}(s/a)$

putting $a=1$, $s=0$, we get

$$\int_0^\infty \frac{\sin t}{t}\,dt = \pi/2 - \tan^{-1}(0/1) = \pi/2 - 0 = \pi/2$$ proved.

☑ find $L\{t^3 e^{-3t}\}$

Ans: Since $L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$

$\therefore L\{t^3\} = \frac{3!}{s^4} = \frac{6}{s^4}$

$\therefore L\{t^3 e^{-3t}\} = \frac{6}{(s+3)^4}$

$\because L\{e^{at}F(t)\} = f(s-a)$

Show that $L\{(1+te^{-t})^3\} = \frac{1}{s} + \frac{3}{(s+1)^2} + \frac{6}{(s+2)^3} + \frac{6}{(s+3)^4}$

Ans: $L\{(1+te^{-t})^3\} = L\{1 + t^3e^{-3t} + 3te^{-t} + 3t^2e^{-2t}\}$

$= L\{1\} + L\{t^3e^{-3t}\} + 3L\{te^{-t}\} + 3L\{t^2e^{-2t}\}$

$= \frac{1}{s} + \frac{3!}{(s+3)^4} + 3\cdot\frac{1}{(s+1)^2} + 3\cdot\frac{2!}{(s+2)^3}$

$= \frac{1}{s} + \frac{3}{(s+1)^2} + \frac{6}{(s+2)^3} + \frac{6}{(s+3)^4}$

If $L\{F(t)\} = f(s)$

$L\{e^{at}f(t)\} = f(s-a)$

$L\{t^n\} = \frac{n!}{s^{n+1}}$

$L\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$

Find $L\{t^2\cos at\}$ Using the rel^n: $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$

$\therefore L\{t^2\cos at\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{s}{s^2+a^2}\right)$

$= (-1)^2 \frac{d}{ds}\left\{\frac{(s^2+a^2)\cdot 1 - s(2s)}{(s^2+a^2)^2}\right\}$

$(f(t) \Longrightarrow \cos at)$

$\because L\{\cos at\} = \frac{s}{s^2+a^2}$

$= + \frac{(s^2+a^2)^2\{2s-4s\} - (a^2-s^2)\cdot(2)(s^2+a^2)\cdot(2s)}{(s^2+a^2)^4}$

$= \frac{-2s(s^2+a^2)^2 - 4s(a^2-s^2)(s^2+a^2)}{(s^2+a^2)^4}$

$= \frac{-2s(s^2+a^2) - 4s(a^2-s^2)}{(s^2+a^2)^3}$

$= \frac{2s^3 - 6a^2s^2}{(s^2+a^2)^3}$

$= \frac{2s(s^2-3a^2)}{(s^2+a^2)^3}$