Solve the following problems.

1. In the ring Z_{20}, prove that $I = \{\bar{n} : n \text{ is even}\}$ is an ideal.

2. In the ring $Z[i]$, show that
$I = \{a + bi : a, b \in Z \text{ and } a, b \text{ are even}\}$
is an ideal.

3. If I_1 is an ideal of the ring R_1 and I_2 is an ideal of the ring R_2, prove that $I_1 \times I_2$ is an ideal of $R_1 \times R_2$.

4. Find all ideals of the Cartesial product $F_1 \times F_2$ of two fields F_1 and F_2.

5. Let I be a ideal of a commutative ring R. Define the <u>annihilator</u> of I to be the set $\text{ann}\, I = \{r \in R : ra = 0, \forall a \in I\}$. Prove that $\text{ann}\, I$ is an ideal of R.

6. Let $I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(Z) : a, b, c, d \text{ are even integers} \right\}$

Show that I is an ideal of $M_2(Z)$. Find the number of elements of the quotient ring $M_2(Z)/I$.

7. Show that the set $12\mathbb{Z}$ is an ideal of the ring $3\mathbb{Z}$. Describe the quotient ring $3\mathbb{Z}/12\mathbb{Z}$.

8. (a) Let p be a prime integer and let T be the set of rational numbers (in lowest terms) whose denominators are not divisible by p. Prove that T is a ring

(b) Let I be the set of elements of T whose numerators are divisible by p. Prove that I is an ideal in T.

(c) Show that T/I consists of exactly p distinct elements.

9. Let I be an ideal in a commutative ring R and let

$$J = \{r \in R : r^n \in I \text{ for some positive integer } n\}$$

Prove that J is an ideal that contains I.

10. If I is an ideal in a commutative ring R with unity and if $a \in R$ with $a \notin I$, prove that the set $J = \{m + ra : r \in R, m \in I\}$ is an ideal such that $I \subsetneq J$.