Find the Laplace transform of (i) $\sin\sqrt{t}$ (ii) $\cos\sqrt{t}/\sqrt{t}$

Ans: (i) we have

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

$$\therefore \sin\sqrt{t} = \sqrt{t} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \frac{t^{7/2}}{7!} + \cdots$$

Using the relation

$$L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$$

$$L\{\sin\sqrt{t}\} = \frac{\Gamma(3/2)}{s^{3/2}} - \frac{\Gamma(5/2)}{s^{5/2}.3!} + \frac{\Gamma(7/2)}{s^{7/2}.5!} - \frac{\Gamma(9/2)}{s^{9/2}.7!} + \cdots$$

$$= \frac{\sqrt{\pi}}{s^{3/2}}\left[\frac{1}{2} - \left(\frac{1}{3!}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{1}{s}\right) + \left(\frac{1}{5!}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{1}{s^2}\right) - \left(\frac{1}{7!}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{1}{s^3}\right) + \cdots\right]$$

$$= \frac{1}{2s}\left(\frac{\pi}{s}\right)^{1/2}\left[1 - \frac{1}{(2^2s)} + \frac{1}{(2^2s)^2\,2!} - \frac{1}{(2^2s)^3\,3!} + \cdots\right]$$

$$= \frac{1}{2s}\left(\frac{\pi}{s}\right)^{1/2}\left[e^{-\frac{1}{2^2s}}\right]$$

$$= \frac{1}{2s}\left(\frac{\pi}{s}\right)^{1/2} e^{-1/4s}$$

(ii) Let $L\{F(t)\} = L\{\sin\sqrt{t}\} = \frac{1}{2s}\left(\frac{\pi}{s}\right)^{1/2} e^{-1/4s} = f(s)$, say

$$F(0) = \sin 0 = 0$$

$$F'(t) = \frac{d}{dt}(\sin\sqrt{t}) = \frac{1}{2\sqrt{t}}\cos\sqrt{t} = \frac{1}{2}\frac{\cos\sqrt{t}}{\sqrt{t}}$$

We have $L\{F'(t)\} = sf(s) - F(0)$

$$= sf(s) - 0 = sf(s)$$

$$\therefore \frac{1}{2}L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \frac{1}{2}\left(\frac{\pi}{s}\right)^{1/2} e^{-1/4s}$$

Or, $L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = (\pi/s)^{1/2} e^{-1/4s}$ **Ans.**

Prob:- find $L\left\{\int_0^t \frac{\sin x}{x}\,dx\right\}$

Ans: We know that if $L\{F(t)\} = f(s)$ then

$$\frac{1}{s} f(s) = L\left\{\int_0^t f(u)\,du\right\} \longrightarrow (1) \quad \text{[from page (7)]}$$

and if $L\{f(t)\} = f(s)$ then $L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(x)\,dx \longrightarrow (2)$ (from page 14)

Now $L\{\sin at\} = \frac{a}{s^2+a^2}$ hence $L\{\sin t\} = \frac{1}{1+s^2}$

$$\therefore L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty L(\sin t)\,ds \quad \text{(using 2nd relⁿ)}$$

$$= \int_s^\infty \frac{ds}{1+s^2} = \left(\tan^{-1} s\right)_s^\infty = \frac{\pi}{2} - \tan^{-1}(s) = \cot^{-1} s \longrightarrow (3)$$

$$\therefore \boxed{L\left\{\int_0^t \frac{\sin t}{t}\,dt\right\} = \frac{1}{s}\cot^{-1} s} \quad \text{(using relⁿ (1) \& (3))}$$

Problem Show that $\int_0^\infty \frac{F(t)}{t}\,dt = \int_0^\infty f(x)\,dx$, assuming that the integrals converge and $L\{F(t)\} = f(s)$ and hence prove that $\int_0^\infty \frac{\sin t}{t}\,dt = \pi/2$

Ans:- We know that $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty f(x)\,dx$

or, $\int_0^\infty e^{-st}\frac{f(t)}{t}\,dt = \int_s^\infty f(x)\,dx$

$= \int_s^0 f(x)\,dx + \int_0^\infty f(x)\,dx$

Making $s \to 0$, we get

$$\int_0^\infty 1.\, f(t)/t\,dt = 0 + \int_0^\infty f(x)\,dx$$

or $$\boxed{\int_0^\infty \frac{f(t)}{t}\,dt = \int_0^\infty f(x)\,dx}$$

2nd part: Since $L\{\sin t\} = \frac{1}{1+s^2}$

Take $f(t) = \sin t$ and $f(s) = \frac{1}{1+s^2}$

$\therefore$ Using the above relation

$$\int_0^\infty \frac{F(t)}{t}\,dt = \int_0^\infty f(x)\,dx$$

or $$\int_0^\infty \frac{\sin t}{t}\,dt = \int_0^\infty \frac{dx}{1+x^2} = (\tan^{-1}x)_0^\infty = \pi/2 - 0 = \pi/2$$

$$\therefore \boxed{\int_0^\infty \frac{\sin t}{t}\,dt = \pi/2}$$

Prob:- Find the Laplace transform of $F(t)$ where

$$F(t) = \begin{cases} t/T & 0 < t < T \\ 1 & t \geq T \end{cases}$$

Ans:

$$L\{f(t)\} = \int_0^\infty e^{-st} F(t)\, dt$$

$$= \int_0^T e^{-st} f(t)\, dt + \int_T^\infty e^{-st} F(t)\, dt$$

$$= \int_0^T \frac{t}{T} e^{-st}\, dt + \int_T^\infty e^{-st} \cdot 1 \cdot dt$$

$$= \frac{1}{T}\left[t \cdot \frac{e^{-st}}{(-s)} - 1 \cdot \frac{e^{-st}}{(-s)^2} \right]_0^T - \frac{1}{s}\left(e^{-st}\right)_T^\infty$$

$$= \frac{1}{T}\left[\left(\frac{T e^{-sT}}{-s} - \frac{e^{-sT}}{s^2} \right) - \left(0 - \frac{1}{s^2} \right) \right] + \frac{e^{-sT}}{s}$$

$$= \frac{e^{-sT}}{Ts} - \frac{e^{-sT}}{Ts^2} + \frac{1}{Ts^2} + \frac{e^{-sT}}{s} = \frac{1}{Ts^2} - \frac{e^{-sT}}{Ts^2}$$

$$= \frac{1}{Ts^2}\left(1 - e^{-sT}\right)$$

Exercise: (1) prove that $L\{H(t)\} = \dfrac{2(1-e^{-\pi s})}{s^2+4}$

where $H(t) = \begin{cases} \sin 2t, & 0 < t < \pi \\ 0, & t \geq \pi \end{cases}$

Home Work

Prob: Show that $L\{F(t)\} = \left(\frac{1}{s^2}+\frac{1}{s}\right)e^{-s} - \left(\frac{1}{s^2}+\frac{2}{s}\right)e^{-2s}$ where

$$F(t) = \begin{cases} 0 & ; \ 0 < t \leq 1 \\ t & ; \ 1 \leq t \leq 2 \\ 0 & ; \ t > 2 \end{cases}$$

Ans: $L\{F(t)\} = \int_0^\infty e^{-st} F(t)\,dt$

$$= \int_0^1 e^{-st}(0)\,dt + \int_1^2 e^{-st}\,t\,dt + \int_2^\infty e^{-st}(0)\,dt$$

$$= 0 + \left[t\,\frac{e^{-st}}{(-s)} - 1\cdot\frac{e^{-st}}{(-s)^2}\right]_1^2 + 0$$

$$= \left(\frac{2e^{-2s}}{-s} - \frac{e^{-2s}}{s^2}\right) - \left(\frac{1\cdot e^{-s}}{-s} - \frac{e^{-s}}{s^2}\right)$$

$$= \left(\frac{1}{s^2}+\frac{1}{s}\right)e^{-s} - \left(\frac{1}{s^2}+\frac{2}{s}\right)e^{-s}$$

Exercise (2)

Find the Laplace transform of $F(t)$ where

$$F(t) = \begin{cases} t & , \ 0 \leq t \leq \frac{1}{2} \\ (t-1) & , \ \frac{1}{2} \leq t \leq 1 \\ 0 & , \ t > 1 \end{cases}$$

Home work

$$\text{Ans} = \frac{1}{s^2}\left(1 - s\,e^{-s/2} - e^{-s}\right)$$

Laplace transform of Dirac delta function

Def. $\delta(t) = \begin{cases} \infty, & t=0 \\ 0, & t\neq 0. \end{cases}$ and $\int_{-\infty}^{\infty} \delta(t)\,dt = 1 \longrightarrow (1)$

or $\delta(t-a) = \begin{cases} \infty, & t=a \\ 0, & t\neq a \end{cases}$ and $\int_{-\infty}^{\infty} \delta(t-a)\,dt = 1.$ $\longrightarrow (2)$

$$L\{\delta(t)\} = \int_0^{\infty} e^{-st}\,\delta(t)\,dt = 1 \longrightarrow (3)$$

and $$L\{\delta(t-a)\} = \int_0^{\infty} e^{-st}\,\delta(t-a)\,dt = e^{-as} \longrightarrow (4)$$

$(a>0)$

Notice that the two formulae are consistent if we put $a=0$ then eqn. (3) and (4) will be the same.

Eqn. (4) is valid for $a>0$, if $a<0$ then

$$L\{\delta(t-a)\} = 0 \quad \text{for } a<0$$

because in this range $(0,\infty)$, $\delta(t-a)=0$ when 'a' is negative.

As an alternative method, $\delta(t)$ may be considered the limit as $\epsilon \to 0$ of $f(t)$, where

$$f(t) = \begin{cases} 0, & t<0 \\ \frac{1}{\epsilon}, & 0<t<\epsilon \\ 0, & t>\epsilon \end{cases}$$

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$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\,dt = \int_0^{\epsilon} \frac{e^{-st}}{\epsilon}\,dt + 0 = \frac{1}{\epsilon}\left(\frac{e^{-st}}{-s}\right)_0^{\epsilon}$$

$$= \frac{1}{\epsilon s}\left(1 - e^{-\epsilon s}\right)$$

Taking the limit of the integral (instead of the integral of the limit), we have

$$\lim_{\epsilon \to 0} L\{F(t)\} = 1.$$

or $$L\{\delta(t)\} = 1.$$

$$\frac{1}{\epsilon s}(1 - e^{-\epsilon s}) \qquad a = \epsilon s$$
$$= \frac{1}{a}[1 - e^{-a}]$$
$$= \frac{1}{a}\left[1 - \left\{1 - a + \frac{a^2}{2!} - \frac{a^3}{3!} + \cdots\right\}\right]$$
$$= 1, \text{ neglecting h.o.t as 'a' is small}$$

This δ-fn is frequently called the impulse function because it is so useful in describing impulsive forces, i.e, forces lasting only a short time.

Exercise (3)

Find the Laplace transform of the square wave (period a) defined by

$$F(t) = \begin{cases} 1, & 0 < t < a/2 \\ 0, & \frac{a}{2} < t < a \end{cases}$$

Ans $$f(s) = \frac{1}{s} \cdot \frac{1 - e^{-as/2}}{1 - e^{-as}}$$

(Home work)

Unit step function

The unit step functions $u(t-a)$ is defined as

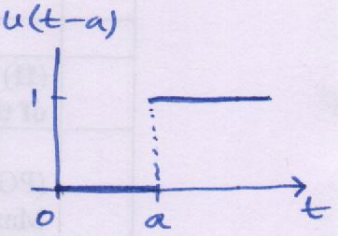
$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

This is also known as Heaviside's unit step function.

Prob: Find Laplace transform of $F(t-a)$ where $F(t-a)$ is Heaviside's unit step function.

Sol: We have

$$F(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$



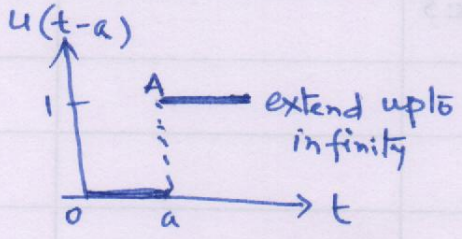
$$\therefore L\{F(t-a)\} = \int_0^\infty e^{-st} F(t-a)\,dt$$

$$= \int_0^a e^{-st} \cdot (0)\,dt + \int_a^\infty e^{-st} (1)\,dt$$

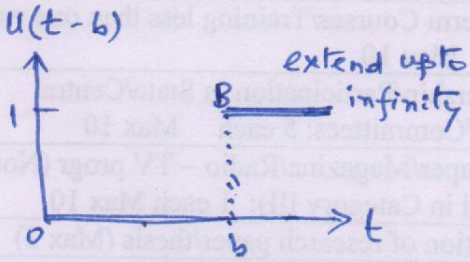
$$= 0 + \left(\frac{e^{-st}}{-s}\right)_a^\infty$$

$$= \frac{e^{-sa}}{s}$$

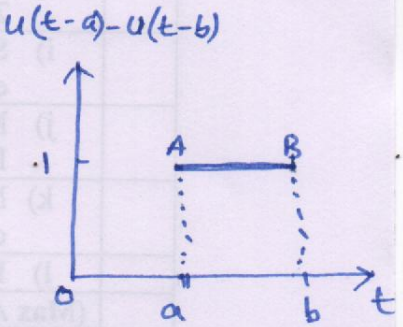
Draw the graphs of $u(t-a)$, $u(t-b)$ and $u(t-a)-u(t-b)$ where $(b>a)$.



$u(t-a)$ indicates a straight line from A to infinity



$u(t-b)$ is a straight line from B to infinity



The graph of $u(t-a)-u(t-b)$ is AB.

☑ Express the function

$$f(t) = \begin{cases} V, & a < t < b \\ 0, & t > b \end{cases}$$

interms of unit step function and find its Laplace transform.

Ans:

$$f(t) = \begin{cases} V, & a < t < b \\ 0, & t > b \end{cases}$$

$$= V \begin{cases} 1, & a < t < b \\ 0, & t > b \end{cases}$$

$$= V[u(t-a) - u(t-b)]$$

Now using the relation of Laplace transform of unit step function

$$L\{f(t)\} = V\,L\{u(t-a)\} - V\,L\{u(t-b)\}$$

$$= V \cdot \frac{e^{-sa}}{s} - V \cdot \frac{e^{-sb}}{s}$$

$$= \frac{V}{s}\left(e^{-sa} - e^{-sb}\right)$$

☑ Express the following function interms of unit step function

$$f(t) = \begin{cases} t-1, & 1 < t < 2 \\ 3-t, & 2 < t < 3 \end{cases}$$

and find its Laplace transform.

Ans:

$$f(t) = \begin{cases} t-1 & 1 < t < 2 \\ 3-t & 2 < t < 3 \end{cases}$$

$$= (t-1)\{u(t-1) - u(t-2)\} + (3-t)\{u(t-2) - u(t-3)\}$$

$$= (t-1)u(t-1) - 2(t-2)u(t-2) + (t-3)u(t-3) \quad \text{[by simplifying]}$$

$$\therefore L\{f(t)\} = L\{(t-1)u(t-1)\} - 2L\{(t-2)u(t-2)\} + L\{(t-3)u(t-3)\}$$

Now

$$L\{(t-1)u(t-1)\}$$

$$= \int_0^\infty e^{-st} \cdot (t-1)u(t-1)\,dt$$

$$= \int_0^1 e^{-st}(t-1)(0)\,dt + \int_1^\infty e^{-st}(t-1)\cdot 1\cdot dt$$

$$= 0 + \left[(t-1)\frac{e^{-st}}{(-s)} - 1\cdot\frac{e^{-st}}{(-s)^2}\right]_1^\infty$$

$$= \frac{e^{-s}}{s^2}$$

Similarly, $L\{(t-2)u(t-2)\} = \frac{e^{-2s}}{s^2}$

$$L\{(t-3)u(t-3)\} = \frac{e^{-3s}}{s^2}$$

Using these results,

$$L\{f(t)\} = \frac{e^{-s}}{s^2} - \frac{2e^{-2s}}{s^2} + \frac{e^{-3s}}{s^2}$$

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