Q. 1 *Prove that the series*

$$1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{5}}-\frac{1}{\sqrt{6}}+\ \ldots$$

Converges and that the series

$$1+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{5}}+\frac{1}{\sqrt{7}}-\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{9}}+\frac{1}{\sqrt{11}}-\frac{1}{\sqrt{6}}+\ \ldots,$$

Consisting of the same learns written in another order, diverges.

The first series converges conditionally by Leibniz's theorem

The second can be written in the form

$$\left(1+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{5}}+\frac{1}{\sqrt{7}}-\frac{1}{\sqrt{4}}\right)+\ \ldots.$$

$$+\left(\frac{1}{\sqrt{4n-3}}+\frac{1}{\sqrt{4n-1}}-\frac{1}{\sqrt{2n}}\right)+\ \ldots..$$

Now,

$$\frac{1}{\sqrt{4n-3}}+\frac{1}{\sqrt{4n-1}}-\frac{1}{\sqrt{2n}}>\frac{1}{\sqrt{4n}}+\frac{1}{\sqrt{4n}}-\frac{1}{\sqrt{2n}}=\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{2n}}=\frac{\sqrt{2}-1}{\sqrt{2n}}$$

But,

$$\sum_{\pi=1}^{\infty}\frac{\sqrt{2}-1}{\sqrt{2n}}\ \textit{is clearly a divergent seies. Hence our series diverges.}$$

$\boldsymbol{Q}$**. 2.** *Find the sum of the series*

$$\sum_{n=1}^{\infty}\Big(\frac{1}{5^n}+\frac{(-1)^n}{n^2}+\sum_{n=1}^{\infty}\left(\frac{1}{5^{n+1}}+\frac{(-1)^{n+1}}{n^2}\right)$$

Both series clearly converge. Their sum is

$$\sum_{n=1}^{\infty}\left[\frac{1}{5^n}+\frac{1}{5^{n+1}}+\frac{(-1)^n}{n^2}+\frac{(-1)^{n+1}}{n^2}\right]$$

$$=\sum_{n=1}^{\infty}\frac{6}{5^{n+1}}=\frac{6}{5}\sum_{n=1}^{\infty}\frac{1}{5^n}=\frac{6}{5}\ \frac{\frac{1}{5}}{1-\frac{1}{5}}=\frac{3}{10}$$

$\boldsymbol{Q.3.}$ *Test the convergence of*

$$1-\frac{100}{2}+\frac{400}{2^2}-\frac{900}{2^3}+\ \dots+(-1)^n\ \frac{100(n-1)^2}{2^{n-1}}+\ \dots$$

By D'Alembert's test

$$\lim_{n\to\infty}\frac{|u_{n+1}|}{|u_n|}=\lim_{n\to\infty}\frac{100n^2.2^{n-1}}{2^n.100(n-1)^2}=\lim_{n\to\infty}\left(\frac{n}{n-1}\right)^2.\frac{1}{2}=\frac{1}{2}$$

The series coverges absolutely.

$\boldsymbol{Q.4}$ *Prove that the sequence* $u_n=\dfrac{2n-3}{n+2}$ *is (a)monotone (b)bounded.*

(a)*Let us find the difference* $u_{n+1}-u_n$

$$u_{n+1}-u_n=\frac{2(n+1)-3}{n+1+2}-\frac{2n-3}{n+2}$$

$$=\frac{2n^2+4n-n-2-2n^2-6n+3n+9}{(n+3)(n+2)}$$

$$=\frac{7}{(n+3)(n+2)}$$

This differnece is positive for every n, *that is* , $u_{n+1}>u_n$ *for every* n

and hence the sequence is monotone increasing

The boundedness of u_n *can be shown also in the following way*:

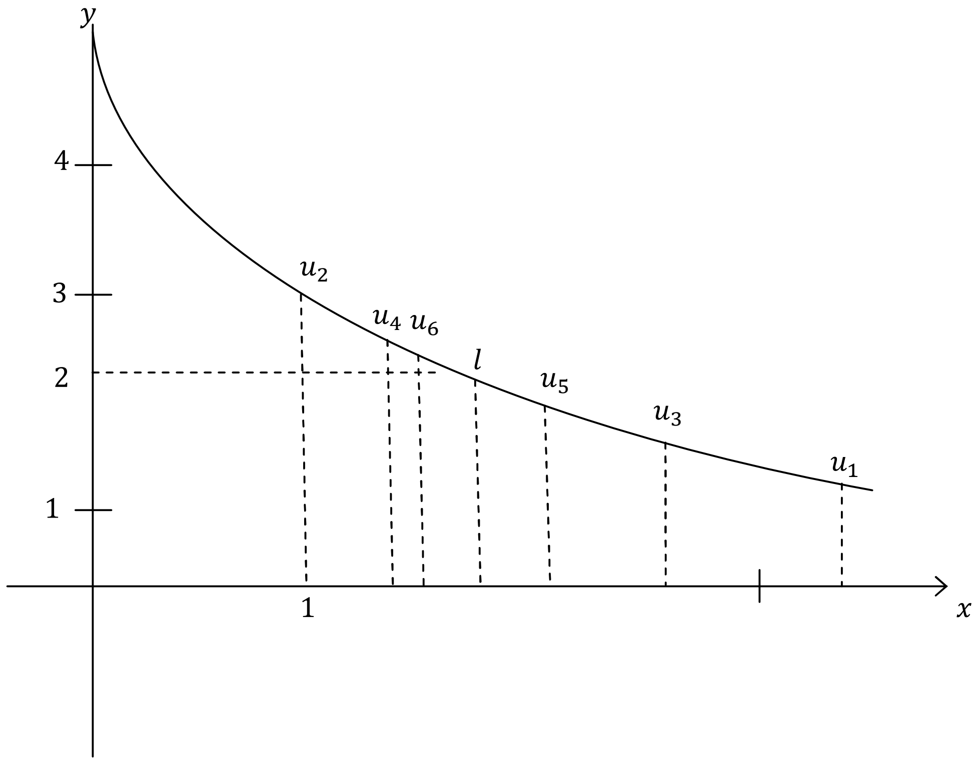
$$u_n=\frac{2n-3}{n+2}<\frac{2n}{n+2}<\frac{2n}{n}=2.$$

$\boldsymbol{Q.5}$ *Given* $u_{n+1}=6/(1+u_n)$ *and* $u_1=1$. *Prove that the sequence converges and find its limit. Find the sequence elements by means of the graph* $y=6/(1+x)$

$$\lim_{n\to\infty}u_{2n+2}=\lim_{n\to\infty}\frac{6}{1+u_{2n+1}}\lim_{n\to\infty}\frac{6}{1+\dfrac{6}{1+u_{2n}}};$$

$$i.e,\quad l=\frac{6}{1+\dfrac{6}{1+l}};\quad l^2+l-6=0;\quad l_1=2,\ \ l_2=-3.$$

Only the positive solution is suitable. The same calculation holds also for the first sequence, so that 2 is the common limit of both subsequence and consequently also of the given sequence. (prove this assertion)



Let us draw the graph of $y = \dfrac{6}{1+x}$ *for* $x > 0$ *. The sequence elements can be found as follows:* $u_1 = 1$ *is the intersection point of the curve* $y = 6/(1+x)$ *and the straight line* $y = 1$. *Take now the ordinate of* u_1 *as an abscissa and obtain* $y_2 = u_2 = 6/(1+u_1) = 3$.

Now take the ordinate of u_2 *as abscissa and obtain* u_2 *on the graph , and so on.*

This construction shows clearly the way u_n *tends to* 2.

Q.6 *Show that* $\sum_{n=1}^{\infty} \frac{1}{n^{2020}}$ *is bounded by* 2.

$$\sum_{n=1}^{\infty} \frac{1}{n^{2020}} < \sum_{n=1}^{\infty} \frac{1}{n^2} < 1 + \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 2$$

Q.7 *Test convergence of the following series*

$$\sum^{\infty} a^n sin^2 n\alpha \, , 0 < a < 1$$

$$t = \lim_{n \to \infty} a \sqrt[n]{sin^2 n\alpha} \leq a < 1$$

Whence the series converges.

Q.8 ***SEE CLASS NOTE***

Q.9 ***SEE CLASS NOTE***

Q.10 ***SEE CLASS NOTE***