(vii) Find the inverse Laplace transform of $\frac{s+2}{s^2-2s+5}$.

Ans: $s^2-2s+5 = (s-1)^2+4 = (s-1)^2+2^2$

$$\therefore L^{-1}\left\{\frac{s+2}{s^2-2s+5}\right\} = L^{-1}\left\{\frac{(s-1)+3}{(s-1)^2+2^2}\right\}$$

$$= L^{-1}\left\{\frac{s-1}{(s-1)^2+2^2} + \frac{3}{(s-1)^2+2^2}\right\}$$

$$= L^{-1}\left\{\frac{s-1}{(s-1)^2+2^2}\right\} + \frac{3}{2} L^{-1}\left\{\frac{2}{(s-1)^2+2^2}\right\}$$

We know that

$$L^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = e^{at}\cos bt$$

and $$L^{-1}\left\{\frac{b}{(s-a)^2+b^2}\right\} = e^{at}\sin bt$$

$$\therefore L^{-1}\left\{\frac{s+2}{s^2-2s+5}\right\} = e^t\cos 2t + \frac{3}{2}e^t \sin 2t$$

(viii) Find the inverse Laplace transform of $\frac{1+2s}{(s+2)^2(s-1)^2}$.

Ans: Suppose $\frac{1+2s}{(s+2)^2(s-1)^2} = \frac{a}{(s+2)} + \frac{b}{(s+2)^2} + \frac{c}{(s-1)} + \frac{d}{(s-1)^2}$

$$= \frac{a(s+2)(s-1)^2 + b(s-1)^2 + c(s+2)^2(s-1) + d(s+2)^2}{(s+2)^2(s-1)^2}$$

The equating the coefficient of s^3, s^2, s and s^0 of both sides of numerators we get

$$a = 0, \quad b = -1/3, \quad c = 0, \quad d = 1/3$$

$$\therefore \quad \frac{1+2s}{(s+2)^2(s-1)^2} = \frac{1}{3}\left[\frac{1}{(s-1)^2} - \frac{1}{(s+2)^2}\right]$$

or $$L^{-1}\left\{\frac{1+2s}{(s+2)^2(s-1)^2}\right\} = \frac{1}{3} L^{-1}\left\{\frac{1}{(s-1)^2} - \frac{1}{(s+2)^2}\right\}$$

$$= \frac{1}{3} L^{-1}\left\{\frac{1}{(s-1)^2}\right\} - \frac{1}{3} L^{-1}\left\{\frac{1}{(s+2)^2}\right\}$$

Now $$L^{-1}\left\{\frac{1}{(s-1)^2}\right\} = -L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s-1}\right)\right\} = -(-1)^1 t^1 e^t = te^t$$

$$\left[\because L^{-1}\{f^n(s)\} = (-1)^n t^n F(t)\right]$$

and $$L^{-1}\left\{\frac{1}{(s+2)^2}\right\} = (-1) L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s+2}\right)\right\} = (-1)(-1)^1 t^1 e^{-2t}$$

$$= te^{-2t}$$

Therefore, $$L^{-1}\left\{\frac{1+2s}{(s+2)^2(s-1)^2}\right\} = \frac{1}{3} t\left(e^t - e^{-2t}\right)$$ Ans.

VII) **Theorem** : (Division by s).

If $L^{-1}\{f(s)\} = F(t)$ then $L^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t F(u)\,du$.

Ans: From the Laplace transform of integral (page-9) we have seen that

$$\frac{1}{s} f(s) = L\left\{\int_0^t F(u)\,du\right\}$$

$$\therefore L^{-1}\left\{\frac{1}{s} f(s)\right\} = \int_0^t F(u)\,du$$

VII) Find the inverse laplace transform of $\frac{1}{s^2(s+1)^2}$.

Ans: $L\{e^{at}\} = \frac{1}{s-a}$ $\therefore L\{e^{-at}\} = \frac{1}{s+a}$ $\therefore L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$

or $L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s+1}\right)\right\} = (-1)^1 \cdot t^1 \cdot e^{-t} = -te^{-t}$

or, $L^{-1}\left\{\frac{1}{(s+1)^2}\right\} = te^{-t}$

or $L^{-1}\left\{\frac{1}{s(s+1)^2}\right\} = \int_0^t x e^{-x}\,dx = \left(x \cdot \frac{e^{-x}}{-1} - 1 \cdot \frac{e^{-x}}{(-1)^2}\right)_0^t$

$= 1 - e^{-t}(t+1)$

or $L^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\} = \int_0^t \left[1 - e^{-x}(x+1)\right] dx$

$= \left[x + e^{-x}(x+1) + e^{-x}\right]_{x=0}^{x=t}$

$= t + e^{-t} + e^{-t}(t+1) - (1+1)$

$= te^{-t} + 2e^{-t} + t - 2$

Show that $L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = \frac{1}{2a^3}(\sin at - at\cos at)$

Ans: we have $L\{\cos at\} = \frac{s}{(s^2+a^2)}$

or, $L\{t\cos at\} = (-1)^1 \cdot \frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)$

$= (-1)\cdot \frac{(s^2+a^2)\cdot 1 - s(2s)}{(s^2+a^2)^2}$

$= -\frac{a^2-s^2}{(s^2+a^2)^2} = \frac{s^2-a^2}{(s^2+a^2)^2}$

$\therefore L\{\sin at - at\cos at\} = \frac{a}{s^2+a^2} - \frac{a(s^2-a^2)}{(s^2+a^2)^2}$

$= \frac{a(s^2+a^2) - a(s^2-a^2)}{(s^2+a^2)^2}$

$= \frac{2a^3}{(s^2+a^2)^2}$

or, $\frac{1}{2a^3} L\{\sin at - at\cos at\} = \frac{1}{(s^2+a^2)^2}$

or, $L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = \frac{1}{2a^3}(\sin at - at\cos at)$

Proved.

VII Prove that the inverse Laplace transform of $\frac{1}{s^3(s^2+1)}$ is $\left(\frac{t^2}{2} + \cos t - 1\right)$

Ans.

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}$$

$$\therefore L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = L^{-1}\left\{\frac{1}{s^2}\right\} - L^{-1}\left\{\frac{1}{s^2+1}\right\} = t - \sin t$$

$$\therefore L^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} = \int_0^t (x - \sin x)\, dx$$

$$= \frac{t^2}{2} + (\cos x)_{x=0}^{x=t}$$

$$= t^2/2 + \cos t - 1. \quad \text{Ans.}$$

[N.B:- You can prove this relation in another method].