VII Apply the convolution theorem to show that

$$\int_0^t \sin u \cos(t-u)\, du = \frac{t}{2}\sin t$$

See page (35)

$\int_0^t f_1(x) f_2(t-x)\, dx$

$= L^{-1}\{f_1(s) f_2(s)\}$

convolution theorem

**Ans.** Let $f(t) = \int_0^t \sin u \cos(t-u)\, du$

Then by convolution theorem,

$$L\{f(t)\} = L\{\sin t\} \cdot L\{\cos t\}$$

$$= \frac{1}{s^2+1} \cdot \frac{s}{s^2+1}$$

$$= \frac{s}{(s^2+1)^2}$$

$$\therefore F(t) = L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} \longrightarrow (1)$$

Let $g(s) = \frac{1}{s^2+1} = L\{G(t)\} \Rightarrow G(t) = \sin t$

$$g'(s) = \frac{-2s}{(s^2+1)^2}$$

$$\therefore L^{-1}\{g'(s)\} = L^{-1}\left\{\frac{-2s}{(s^2+1)^2}\right\}$$

or, $(-1)^1 \cdot t^1 \cdot \underbrace{L^{-1}\{g(s)\}} = L^{-1}\left\{\frac{-2s}{(s^2+1)^2}\right\}$

or, $\frac{-t}{-2} \cdot \underbrace{\sin t^1} = L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$

or, $L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{t}{2}\sin t \longrightarrow (2)$

From (1) and (2) we see that the required result.

H.W Excercise

(1) Apply convolution theorem to prove that

$$\beta(m,n) = \int_0^1 x^{m-1}\,(1-x)^{n-1}\,dx = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)},$$

$(m>0, n>0)$.

Hence deduce that

$$\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta = \frac{1}{2}\beta(m,n)$$

$$= \frac{\Gamma(m)\,\Gamma(n)}{2\,\Gamma(m+n)}$$

(2) Find (i) $L^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\}$, (ii) $L^{-1}\left\{\frac{1}{s^2(s^2+1)(s^2+9)}\right\}$

(iii) $L^{-1}\left\{s\log\left(\frac{s-1}{s+1}\right)+2\right\}$

Ans.- (i) $t - \sin t$

(ii) $\frac{t}{9} - \frac{1}{8}\sin t + \frac{1}{72}\left(\frac{\sin 3t}{3}\right)$

(iii) $\frac{2}{t^2}\sinh t - \frac{2}{t}\cosh t$

Co-ordinator IQAC Krishnath College Berhampore

## Solution of differential equations by Laplace transform —

Ordinary linear differential equations with constant co-efficient can be easily solved by the Laplace transform method, without finding the general solution and the arbitrary constants.

### Working procedure —

(1) Take the laplace transform of both sides of the differential equation using the formula of laplace transform of derivatives and the given initial conditions.

(2) Transpose the terms with minus signs to the right.

(3) Divide by the co-efficient of $\bar{y}$ (laplace transform of $y$), getting $\bar{y}$ as a known function of $s$.

(4) Resolve this function of $s$ into partial fractions and take the inverse transform of both sides. This gives $y$ as a function of $t$ which is the desired solution satisfying the given conditions.

**Prob:-** Using Laplace transfooms, find the solution of the initial value problem $y'' - 4y' + 4y = 64 \sin 2t$, $y(0) = 0$, $y'(0) = 1$.

**Ans:-** $y'' - 4y' + 4y = 64 \sin 2t$ ⟶ (1)

Taking laplace transform of both sides of equation (1),

see page -(7)

$L\{F'''(t)\} = s^3 f(s) - s^2 F(0) - s F'(0) - F''(0)$

$L\{F''(t)\} = s^2 f(s) - s F(0) - F'(0)$

$L\{F'(t)\} = s f(s) - F(0)$

$$[s^2\bar{y} - sy(0) - y'(0)] - 4[s\bar{y} - y(0)] + 4\bar{y} = 64 \times \left(\frac{2}{s^2+4}\right) \rightarrow (2)$$

Putting $y(0)=0$ and $y'(0)=1$ we get

$$s^2\bar{y} - 1 - 4s\bar{y} + 4\bar{y} = \frac{128}{s^2+4}$$

or, $\bar{y}(s^2 - 4s + 4) = 1 + \frac{128}{s^2+4}$

or, $\bar{y}(s-2)^2 = 1 + \frac{128}{s^2+4}$

or, $\bar{y} = \frac{1}{(s-2)^2} + \frac{128}{(s-2)^2(s^2+4)}$

$$= \frac{1}{(s-2)^2} - \frac{8}{s-2} + \frac{16}{(s-2)^2} + \frac{8s}{s^2+4}$$ (by the method of partial fractions)

$$\therefore y = L^{-1}\left\{-\frac{8}{(s-2)} + \frac{17}{(s-2)^2} + \frac{8s}{s^2+4}\right\}$$

$$= -8e^{2t} + 17te^{2t} + 8\cos 2t$$

## Exercise (H.M)

(1) Using laplace transform find the solution of the initial value problem $y'' + 25y = 10\cos 5t$, $y(0)=2$, $y'(0)=0$.

(2) Using Laplace transform find the solution of initial value problem $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t$ with $x=2$, $\frac{dx}{dt} = -1$ at $t=0$.

Ans – (1) $2\cos 5t + t\sin 5t$

(2) $2e^t - 3te^t + \frac{1}{2}t^2e^t$