1. Force on a charged particle:

For a moving charge Q in the presence of both electric and magnetic fields, the total force on the charge is given by,

$$\vec{F} = \vec{F_e} + \vec{F_m}$$

$$= Q\vec{E} + Q\vec{v}\times\vec{B}$$

$$= Q(\vec{E} + \vec{v}\times\vec{B}) \quad (1)$$

This is known as Lorentz force equation.

2. Force on a current element:

To determine the force on a current element $I\vec{dl}$ of a current carrying conductor due to the magnetic field $\vec{B}$, we use $\vec{F_m} = Q(\vec{v}\times\vec{B})$. (2)

We know,

$$\vec{J} = \rho\vec{v} \quad (3)$$

We also know the relationship between current elements

$$I\vec{dl} = \vec{K}ds = \vec{J}dV \quad (4)$$

Therefore, $I\vec{dl} = (\rho\vec{v})dV = dQ\,\vec{v}$ $\quad [\because dQ = \rho dV]$

Alternatively,

$$I\vec{dl} = \frac{dQ}{dt}\vec{dl} = dQ\frac{\vec{dl}}{dt} = dQ\,\vec{v}$$

$$\therefore \boxed{I\vec{dl} = dQ\vec{v}} \quad (5)$$

Thus the force on a current element $I\vec{dl}$ in a magnetic field $\vec{B}$ is found by using eq (2)

$$\vec{dF_m} = dQ(\vec{v}\times\vec{B}) = I(\vec{dl}\times\vec{B}) \quad (6)$$

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If the current I is through a closed path L or circuit, the force on the circuit is given by

$$\vec{F}_m = \oint_L I\, d\vec{l} \times \vec{B} \qquad (7)$$

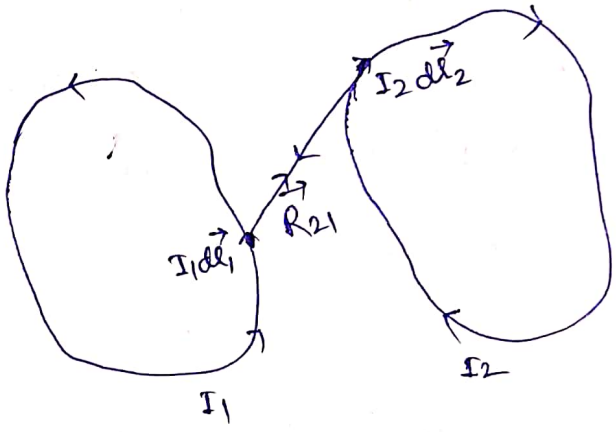
For surface and volume current this equation becomes

$$\vec{F}_m = \int_S \vec{K}\, ds \times \vec{B} = \int_S (\vec{K} \times \vec{B})\, ds \qquad (8a)$$

and $$\vec{F}_m = \int_V \vec{J}\, dV \times \vec{B} = \int_V (\vec{J} \times \vec{B})\, dV \qquad (8b)$$

3. Force between two current Elements:

Let's consider the force between two elements $I_1 d\vec{l}_1$ and $I_2 d\vec{l}_2$. According to Biot-Savart law both current elements produce magnetic fields. So, we may find the force $d(d\vec{F}_1)$ on element $I_1 d\vec{l}_1$ due to the field $d\vec{B}_2$ produced by element $I_2 d\vec{l}_2$ as shown in the fig.



$$d(d\vec{F}_1) = I_1 d\vec{l}_1 \times d\vec{B}_2 \qquad (9)$$

But from Biot-Savart law,

$$d\vec{B}_2 = \frac{\mu_0 I_2\, d\vec{l}_2 \times \hat{R}_{21}}{4\pi R_{21}^2} \qquad (10)$$

Hence, $$d(d\vec{F}_2) = \frac{\mu_0 I_1 d\vec{l}_1 \times (I_2 d\vec{l}_2 \times \hat{R}_{21})}{4\pi R_{21}^2}$$

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Therefore the total force $\vec{F_1}$ on current loop 1 due to current loop2 as shown in the fig is

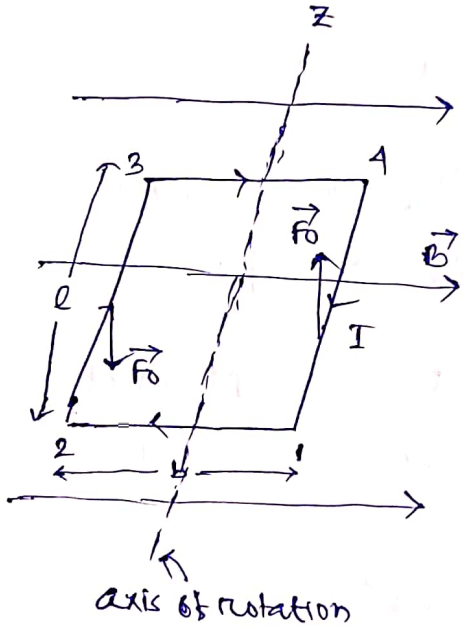
$$\vec{F_1} = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{L1} \oint_{L2} \frac{d\vec{l_1} \times (d\vec{l_2} \times \hat{R}_{21})}{R_{21}^2} \quad \text{and} \quad \vec{F_2} = -\vec{F_1}$$

4. Magnetic torque and moment :

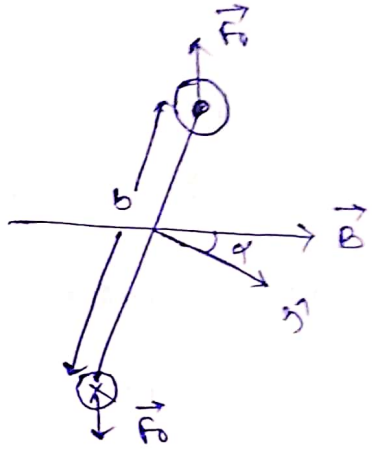
The torque $\vec{\tau}$ on the loop is the vector product of the force $\vec{F}$ and the moment arm $\vec{r}$.

i.e. $$\vec{\tau} = \vec{r} \times \vec{F}$$

Let us apply this to a rectangular loop of length 'l' and breadth 'b' placed in a uniform magnetic field $\vec{B}$ as shown in fig(a). From the figure we notice that $d\vec{l}$ is parallel to $\vec{B}$ along sides 1-2 and 3-4 of the loop and no force is exerted on those sides.



(a)



(b)

Thus, $$\vec{F} = I \int_2^3 d\vec{l} \times \vec{B} + I \int_4^1 d\vec{l} \times \vec{B}$$

$$= I \int_0^l dz\, \hat{z} \times \vec{B} + I \int_l^0 dz\, \hat{z} \times \vec{B}$$

$$= \vec{F_0} + (-\vec{F_0}) = 0$$

where $|\vec{F_0}| = IBl$, because $\vec{B}$ is uniform.

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Thus no force is exerted on the loop as a whole. However, $\vec{F_0}$ and $-\vec{F_0}$ act at different points on the loop, thereby creating a couple. If the normal to the plane of the loop makes an angle α with $\vec{B}$, as shown in fig(b), the torque on the loop is

$$|\vec{\tau}| = |\vec{F_0}| b \sin\alpha$$

or $$\tau = BIlb\sin\alpha$$

But $lb = A$, the area of the loop. Hence

$$\tau = BIA\sin\alpha$$

We define the quantity $\vec{m} = IA\hat{n}$ as the magnetic dipole moment (in Am^2) of the loop.

- The magnetic moment is the product of the current and area of the loop, its direction is normal to the loop

Therefore, $$\vec{\tau} = \vec{m} \times \vec{B}$$

This is the expression to determine the torque on a planar loop of any arbitary shape although it was obtained by using a rectangular loop.

5. A Magnetic Dipole:

A bar magnet or a small filamentary current loop is usually referred to as a magnetic dipole. Let us determine the magnetic field $\vec{B}$ at an observation point $P(r, \theta, \phi)$ due to a circular loop carrying current I as in fig 1.

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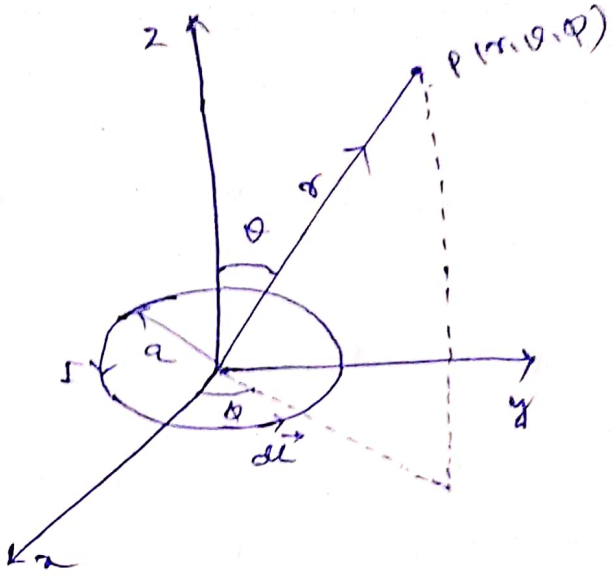
The magnetic vector potential at P is

$$\vec{A} = \frac{\mu_0 I}{4\pi} \oint \frac{d\vec{l}}{r} \qquad (1)$$



It can be shown that at far field ($r >> a$, so that the loop appears small at the observation point), $\vec{A}$ has only ϕ-component and it is given by

$$\vec{A} = \frac{\mu_0 I \pi a^2 \sin\theta}{4\pi r^2} \hat{\phi} \qquad (2)$$

or $$\vec{A} = \frac{\mu_0 \vec{m} \times \hat{r}}{4\pi r^2} \qquad (3)$$

where $\vec{m} = I\pi a^2 \hat{z}$, the magnetic moment of the loop, and $\hat{z} \times \hat{r} = \sin\theta \hat{\phi}$. We determine the magnetic flux density $\vec{B}$ from $\vec{B} = \nabla \times \vec{A}$ as

$$\vec{B} = \nabla \times \left[\frac{\mu_0 I \pi a^2}{4\pi} \cdot \frac{\sin\theta}{r^2} \hat{\phi} \right]$$

$$= \frac{\mu_0 (I \pi a^2)}{4\pi} \nabla \times \frac{\sin\theta}{r^2} \hat{\phi}$$

$$= \frac{\mu_0 m}{4\pi r^3} (2\cos\theta \hat{r} + \sin\theta \hat{\theta}) \quad [\text{Using the form of curl in spherical polar coordinates}]$$

Alternatively,

$$\nabla \times \vec{A} = \nabla \times \left(\frac{\mu_0 \vec{m} \times \vec{r}}{4\pi r^3} \right) = \frac{\mu_0}{4\pi} \nabla \times \frac{(\vec{m} \times \vec{r})}{r^3}$$

$$= \frac{\mu_0}{4\pi} \left[\frac{1}{r^3} \nabla \times (\vec{m} \times \vec{r}) + (\vec{m} \times \vec{r}) \times \nabla \left(\frac{1}{r^3} \right) \right]$$

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$$\vec{B} = \frac{\mu_0}{4\pi}\left[\frac{1}{r^3}\,2\vec{m} + \frac{3\{(\vec{m}\cdot\vec{r})\vec{r} - \vec{m}r^2\}}{r^5}\right]$$

$$= \frac{\mu_0}{4\pi r^3}\left[\frac{3(\vec{m}\cdot\vec{r})\vec{r}}{r^2} - \vec{m}\right]$$

6. Magnetization in materials:

Def: The magnetization $\vec{M}$ (in amp/m) is the magnetic dipole moment per unit volume.

- If there are N atoms in a given volume ΔV and the kth atom has a magnetic moment $\vec{m}_k$,

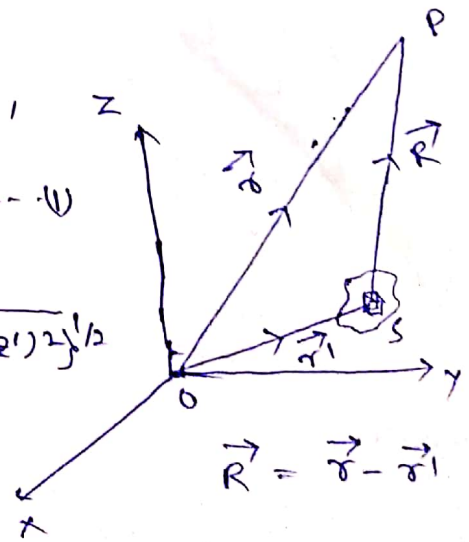
$$\vec{M} = \lim_{\Delta V \to 0} \frac{\sum_{k=1}^{N} \vec{m}_k}{\Delta V}$$

- Magnetic vector potential due to magnetization:

A medium for which $\vec{M}$ is not zero every where is said be magnetized. For a differential volume dV' the magnetic moment is $d\vec{m} = \vec{M}\,dV'$

Therefore the magnetic vector potential due to $d\vec{m}$ is

$$d\vec{A} = \frac{\mu_0\, d\vec{m} \times \vec{R}}{4\pi R^3} = \frac{\mu_0 \vec{M} \times \vec{R}}{4\pi R^3}\,dV' \quad \text{--- (1)}$$



$\vec{R} = \vec{r} - \vec{r}'$

Now,

$$\nabla' \frac{1}{R} = \sum \hat{i} \frac{\partial}{\partial x'} \frac{1}{\{(x-x')^2 + (y-y')^2 + (z-z')^2\}^{1/2}}$$

$$= \sum \hat{i} \frac{(x-x')}{\{(x-x')^2 + (y-y')^2 + (z-z')^2\}^{3/2}}$$

$$= \frac{\vec{R}}{R^3} \quad (2)$$

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Using eq (2) in eq (1), we get

$$\vec{A} = \frac{\mu_0}{4\pi} \int \vec{M} \times \nabla' \left(\frac{1}{R}\right) dV' \qquad (3)$$

Now,

$$\nabla' \times \left(\frac{\vec{M}}{R}\right) = \frac{1}{R} \nabla' \times \vec{M} - \vec{M} \times \nabla' \left(\frac{1}{R}\right)$$

$$\Rightarrow \vec{M} \times \nabla' \left(\frac{1}{R}\right) = \frac{1}{R} \nabla' \times \vec{M} - \nabla' \times \left(\frac{\vec{M}}{R}\right) \quad \cdots (4)$$

Using eq (4) in eq (3) we get

$$\vec{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\nabla' \times \vec{M}}{R} dV' - \frac{\mu_0}{4\pi} \int_{V'} \nabla \times \left(\frac{\vec{M}}{R}\right) dV' \qquad (5)$$

Applying the vector identity

$$\int \nabla' \times \vec{F} \, dV' = - \oint \vec{F} \times \hat{n} \, ds' \quad [\hat{n} \text{ is a unit vector normal to the surface}]$$

to the second integral of eq (5) we get

$$\vec{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\nabla' \times \vec{M}}{R} dV' + \frac{\mu_0}{4\pi} \oint_{S'} \frac{\vec{M} \times \hat{n}}{R} ds'$$

$$= \frac{\mu_0}{4\pi} \int_{V'} \frac{\vec{J}_b \, dV'}{R} + \frac{\mu_0}{4\pi} \oint_{S'} \frac{\vec{K}_b \, ds'}{R} \qquad (6)$$

where, $\vec{J}_b = \nabla \times \vec{M}$ is called bound volume current density

and $\vec{K}_b = \vec{M} \times \hat{n}$ is called bound surface current density

7. Ampere's law in Magnetized Materials:

The total current density of a magnetized object comes from two contributions (i) bound current density due to magnetization, (ii) free current density due to flow of charges. So,

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$$\vec{J} = \vec{J}_b + \vec{J}_f \qquad (1)$$

Now, from Ampere's law

$$\frac{1}{\mu_0}(\nabla \times \vec{B}) = \vec{J} = \vec{J}_f + \vec{J}_b = \vec{J}_f + (\nabla \times \vec{M})$$

$$\rightarrow \nabla \times \left(\frac{1}{\mu_0}\vec{B} - \vec{M}\right) = \vec{J}_f \qquad (2)$$

The quantity in the first bracket is denoted by the letter $\vec{H}$.

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M} \qquad (3)$$

So, in terms of $\vec{H}$, the Ampere's law becomes

$$\nabla \times \vec{H} = \vec{J}_f \qquad (4)$$

or, in integral form

$$\oint \vec{H} \cdot d\vec{l} = I_{free} \qquad (5)$$

where I_{free} is the total free current passing through the Amperian Loop.

- Relation between magnetic susceptibility and relative permeability:

From eq (3), we get

$$\vec{B} = \mu_0(\vec{H} + \vec{M}) \qquad (6)$$

The relationship in eq (6) holds for all materials whether they are linear or not. For linear materials, $\vec{M}$ (in A/m) depends linearly on $\vec{H}$ such that

$$\vec{M} = \chi_m \vec{H} \qquad (7)$$

where χ_m is a dimensionless quantity (ratio of M to H) called magnetic susceptibility of the medium.

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Susceptibility is more or less a measure of how susceptible (or sensitive) the material is to a magnetic field.

putting eq (7) in eq (6) we get

$$\vec{B} = \mu_0 (1 + \chi_m) \vec{H} = \mu \vec{H} \qquad (8)$$

or $$\vec{B} = \mu_0 \mu_r \vec{H} \qquad (9)$$

where $$\mu_r = 1 + \chi_m = \frac{\mu}{\mu_0} \qquad (10)$$

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