Chapter 5 Magnetostatics

5.1 The Lorentz Force Law

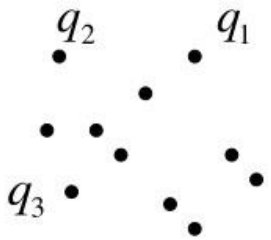
5.2 The Biot-Savart Law

5.3 The Divergence and Curl of $\vec{B}$

5.4 Magnetic Vector Potential

5.1.1 Magnetic Fields

Charges induce electric field

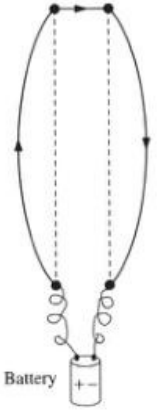


Source charges

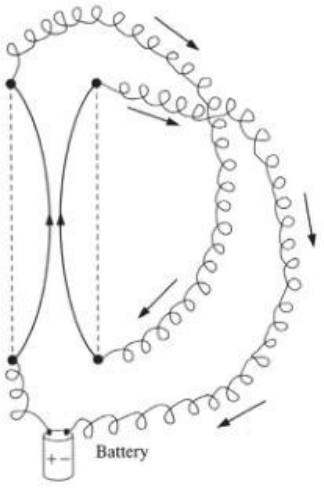
$$Q \longrightarrow \vec{F} \Longrightarrow \vec{E} = \frac{\vec{F}}{Q}$$

Test charge

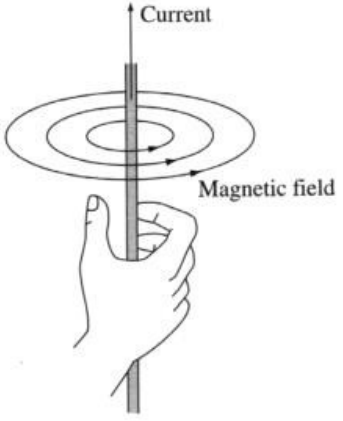
5.1.1

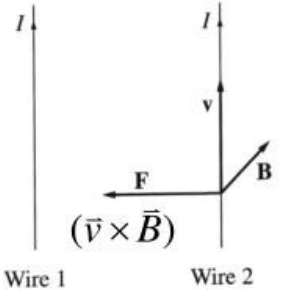


(a) Currents in opposite directions repel.



(b) Currents in same directions attract.



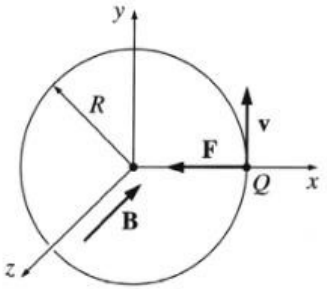


5.1.2 Magnetic Force

$$\vec{F}_{mag} = Q(\vec{v} \times \vec{B})$$

$$\boxed{\vec{F} = Q[\vec{E} + (\vec{v} \times \vec{B})]}$$ Lorentz Force Law

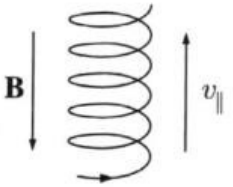
Ex.1 Cyclotron motion



$$\vec{F}_{mag} = QvB$$

$$\| $$

$$\vec{F}_{centripetal} = mvw_c = m\frac{v^2}{R}$$



moment $p = mv = QBR$

cyclotron frequency $w_c = \frac{QB}{m}$

relativistic cyclotron frequency $\boxed{w_c = \frac{QB}{\gamma m}}$

microwave

relativistic electron cyclotron maser $\Leftrightarrow$ light $\rightarrow$ laser

$w \sim w_{ce} = \frac{eB}{\gamma m_e}$

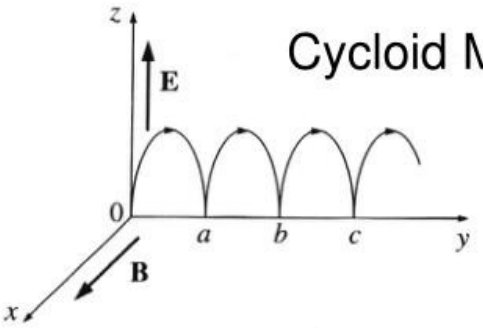
EM wave

$\gamma \approx 1.1$

5.1.2 (3)

Ex.2

Cycloid Motion



$\vec{E}$ v_z

$$\vec{F}_B = qv_z B\hat{y}$$

$$v(t=0)=0 \quad v(t=\Delta t)=v_z = qE\Delta t$$

$$m\frac{d\vec{v}}{dt} = q(\vec{E}+\vec{v}\times\vec{B}) \qquad v_y = \frac{dy}{dt} \quad v_z = \frac{dz}{dt}$$

$$\vec{E} = E\hat{z} \quad \vec{B} = B\hat{x}$$

$$m\frac{d^2y}{dt^2} = q\frac{dz}{dt}B \quad m\frac{d^2z}{dt^2} = q(E-\frac{dy}{dt}B)$$

$$\ddot{y} = w_c\dot{z} \qquad \ddot{z} = w_c(\frac{E}{B}-\dot{y})$$

assume $\dot{y} = V_y + V_{yd} \quad V_{yd} = const$

$\dot{z} = V_z \qquad V_y(t) \quad V_z(t)$

5.1.2 (4)

$$\dot{V}_y = w_c V_z \quad \dot{V}_z = w_c\left(\frac{E}{B} - V_y - V_{yd}\right)$$

$$\text{(x)} \quad (t) \quad \text{(x)}$$

$$\therefore \boxed{V_{yd} = \frac{E}{B}}$$

$$\dot{V}_y = w_c V_z \quad V_y = +C_1 \sin w_c t + C_2 \cos w_c t$$

$$\Rightarrow$$

$$\dot{V}_z = -w_c V_y \quad V_z = -C_2 \sin w_c t + C_1 \cos w_c t$$

$$v_y(t=0) = 0 \quad v_y = V_y + V_{yd} \quad \Rightarrow C_2 = -\frac{E}{B}$$

$$v_z(t=0) = 0 \quad v_z = V_z \quad \Rightarrow C_1 = 0$$

$$v_y = \frac{E}{B}(1 - \cos w_c t)$$

$$v_z = \frac{E}{B}\sin w_c t$$

$$y = \frac{E}{wB}(wt - \sin w_c t) + C_3$$

$$z = -\frac{E}{wB}\cos w_c t + C_4$$

5.1.2 (5)

$$y(t=0)=0 \qquad C_3=0$$

$$z(t=0)=0 \qquad C_4=\frac{E}{wB}$$

$$\therefore \quad y=\frac{E}{w_cB}(w_ct-\sin w_ct)=R(w_ct-\sin w_ct)$$

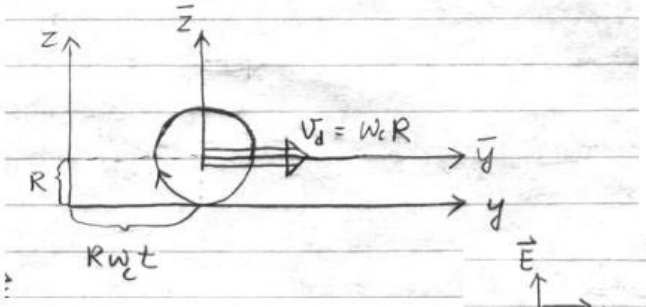
$$z=\frac{E}{w_cB}(1-\cos w_ct) \quad =R(1-\cos w_ct)$$

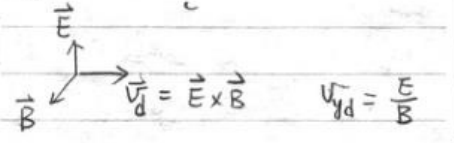
$$\left(y-Rw_ct\right)^2+\left(z-R\right)^2=R^2$$

$$\bar{y}^2+\bar{z}^2=R^2$$

$$\bar{y}=y-Rw_ct \qquad \bar{y}=-R\sin w_ct$$

$$\bar{z}=z-R \qquad \bar{z}=-R\cos w_ct$$





$\vec{E}\times\vec{B}$ *drift.*

the cause of Hall effect

5.1.2 (6)

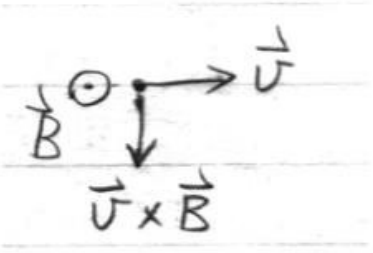
Magnetic forces do not work

for Q moving $\quad d\vec{l} = \vec{v}dt$

$$dW_{mag} = \vec{F}_{mag} \cdot d\vec{l}$$

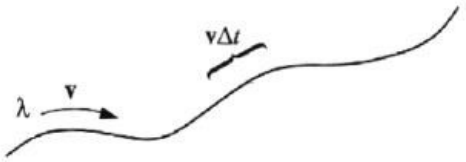
$$= Q(\vec{v} \times \vec{B}) \cdot \vec{v}dt$$

$$= 0$$



5.1.3 Currents

The current in a wire is the charge per unit time passing a given point.



$$I = \frac{(\lambda\, v\Delta t)}{\Delta t} = \lambda\, v \quad (A)$$

Amperes 1A = 1 C/S

$$\vec{I} = \lambda\, \vec{v}$$

The magnetic force on a segment of current-carrying wire

$$\vec{F}_{mag} = \int (\lambda\, dl)\left(\vec{v} \times \vec{B}\right) = \int \left(\vec{I} \times \vec{B}\right) dl$$

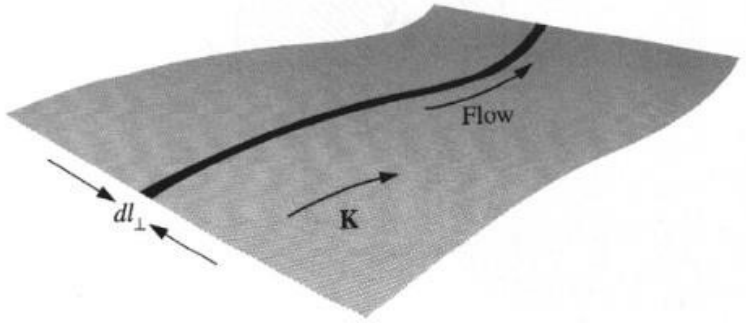
$$= \int I\left(d\vec{l} \times \vec{B}\right) \qquad \vec{I} = I d\vec{l}$$

$$\vec{F}_{mag} = I \int \left(d\vec{l} \times \vec{B}\right)$$

5.1.3 (2)

surface current density

$$\vec{K} = \frac{d\vec{I}}{dl_{\perp}}$$



the current per unit length-perpendicular-to-flow

$$\vec{K} = \sigma\,\vec{v} \qquad \sigma : \textit{surface charge density}$$

(mobile)

The magnetic force on a surface current is

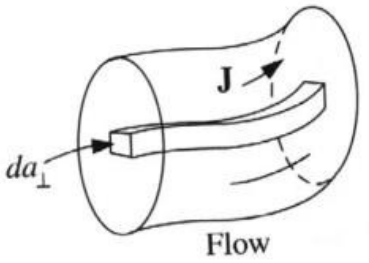
$$\vec{F}_{mag} = \int (\sigma\, da)\left(\vec{v} \times \vec{B}\right) = \int \left(\vec{K} \times \vec{B}\right) da$$

$$\left(\sigma\, da = Q\right)$$

5.1.3 (3)

volume current density

$$\vec{J} = \frac{d\vec{I}}{da_{\perp}}$$



The current per unit area-perpendicular-to-flow

$$\vec{J} = \rho\,\vec{v} \quad \rho: \textit{volume charge density}$$

The magnetic force on a volume current is

$$\vec{F}_{mag} = \int (\rho\, d\tau)\,(\vec{v} \times \vec{B})$$

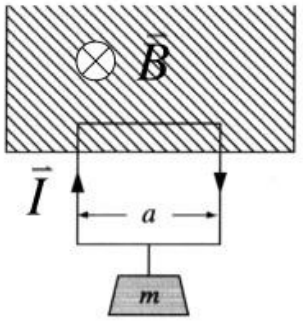
$$(\rho\, d\tau = Q)$$

$$\boxed{\vec{F}_{mag} = \int (\vec{J} \times \vec{B})\, d\tau}$$

5.1.3 (4)

Ex. 3 If $\vec{F}_{mag}(\uparrow) + \vec{F}_g(\downarrow) = 0$

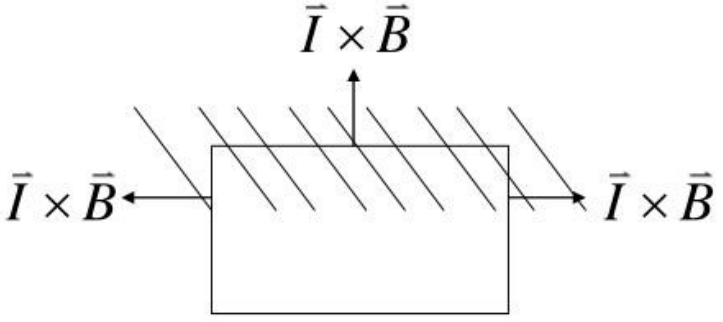
what is $\vec{I}$?



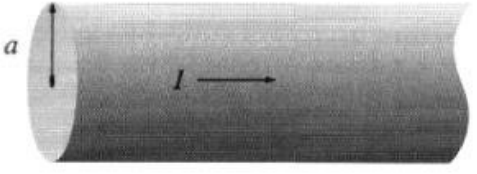
Sol.

$$F_{mag} = IBa = mg$$

$$I = \frac{mg}{Ba}$$



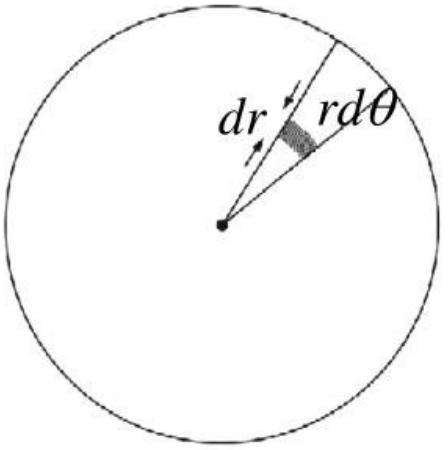
Ex. 4 (a) what is J ?
(uniform I)



Sol. $$J = \frac{I}{\pi R^2}$$

5.1.3 (5)

(b) For $J = kr$, find the total current in the wire.



Sol.
$$I = \int J da = \int (kr)(r dr d\theta)$$
$$= 2\pi k \int_0^R r^2 dr = \frac{2\pi k R^3}{3}$$

$$\sum_{i=1}^{n} (\ ) q_i \vec{v}_i \sim \int_{line} (\ ) \vec{I} dl \sim \int_{surface} (\ ) \vec{K} da \sim \int_{volume} (\ ) \vec{J} d\tau$$

$$q \sim \lambda dl \sim \sigma da \sim \rho d\tau$$

5.1.3 (6)

$J \xleftrightarrow{\text{relation?}} \rho$

$$I = \int_s J da_\perp = \int_s \vec{J} \cdot d\vec{a}$$

$$\oint_s \vec{J} \cdot d\vec{a} = \int_V \left(\nabla \cdot \vec{J}\right) d\tau$$

$$= -\frac{d}{dt}\int_V \rho \, d\tau = -\int_V \left(\frac{\partial \rho}{\partial t}\right) d\tau$$

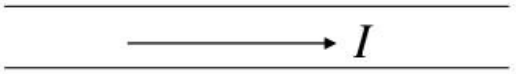
(charge conservation)

Continuity equation

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

5.2.1 Steady Currents

Stationary charges $\Rightarrow$ constant electric field: electrostatics
Steady currents $\Rightarrow$ constant magnetic field: magnetostatics



No time dependence

$$\frac{\partial \rho}{\partial t} = 0 \quad \Rightarrow \quad \nabla \cdot \vec{J} = 0$$

5.2.2 The Magnetic Field of a Steady Current

Biot-Savart Law: for a steady line current

$$\vec{B}(p) = \frac{\mu_0}{4\pi}\int \frac{\vec{I}\times\vec{R}}{R^2}dl = \frac{\mu_0}{4\pi} I\int \frac{d\vec{l}\times\vec{R}}{R^2}$$

$tesla\ (T)$

$1\ tesla = 1\ \frac{N}{A\cdot m}$

$cgs: gauss$

$1\,T = 10^4\ gauss$

Permeability of free space $\mu_0 = 4\pi\times 10^{-7}\ \frac{N}{A^2}$

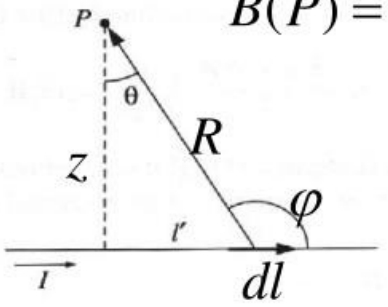
Biot-Savart Law for surface currents $\vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{K}\times\vec{R}}{R^2}da$

Biot-Savart Law for volume currents $\vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}\times\vec{R}}{R^2}d\tau$

for a moving point charge $\vec{B} = \frac{\mu_0}{4\pi} q\frac{\vec{v}\times\vec{R}}{R^2}$ *not steady current*

5.2.2 (2)

$B(P) = ?$



Solution:

$$d\vec{l} \times \hat{R} = dl \sin\varphi = dl \cos\theta \odot$$

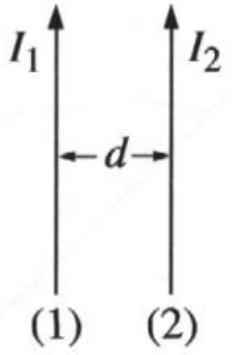
$$l = z\tan\theta \quad dl = z\,d\tan\theta = \frac{z}{\cos^2\theta}d\theta\,,\ \frac{z}{R} = \cos\theta,\ \frac{1}{R^2} = \frac{\cos^2\theta}{z^2}$$

$$B = \frac{\mu_0}{4\pi} I \int \frac{d\vec{l} \times \vec{R}}{R^2} = \frac{\mu_0}{4\pi} I \int_{\theta_1}^{\theta_2} (\frac{\cos^2\theta}{z^2}) \cos\theta \frac{z}{\cos^2\theta} d\theta$$

$$= \frac{\mu_0}{4\pi}\frac{I}{z}\int_{\theta_1}^{\theta_2} \cos\theta d\theta = \frac{\mu_0}{4\pi}\frac{I}{z}(\sin\theta_2 - \sin\theta_1)$$

In the case of an infinite wire, $\theta_1 = -\frac{\pi}{2}$, $\theta_2 = \frac{\pi}{2}$; $\quad B = \frac{\mu_0 I}{2\pi z}$

5.2.2 (3)



Force?

The field at (2) due to I_1 is

$$\vec{B} = \frac{\mu_0 I_1}{2\pi d}(\otimes)$$

The force at (2) due to I_1 is

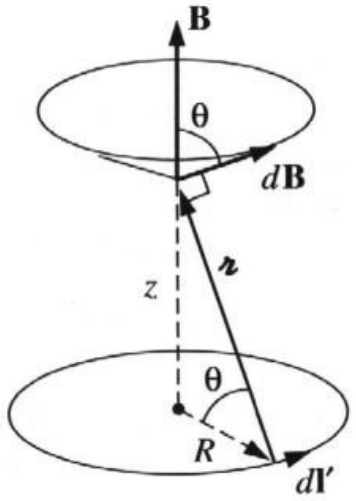
$$\vec{F} = I_2(\frac{\mu_0 I_1}{2\pi d})\int dl$$

The force per unit length is

$$f = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{d}$$

5.2.2 (4)

$B = ?$



$$\vec{B} = \int d\vec{B}\cos\theta\,\hat{z}$$

$$= \frac{\mu_0}{4\pi} I \int \frac{dl}{r^2}\cos\theta\,\hat{z}$$

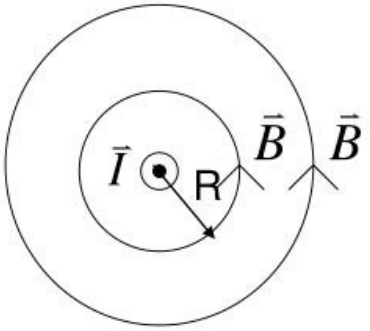
$$= \frac{\mu_0 I}{4\pi}\left(\frac{\cos\theta}{r^2}\right) 2\pi R\hat{z}$$

$$= \frac{\mu_0 I}{2}\left(\frac{\cos\theta}{r^2}\right) R\hat{z}$$

$$\cos\theta = \frac{R}{r} \qquad r = (R^2 + z^2)^{\frac{1}{2}}$$

$$\vec{B} = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}\hat{z}$$

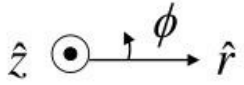
5.3.1 an example: Straight-Line Currents



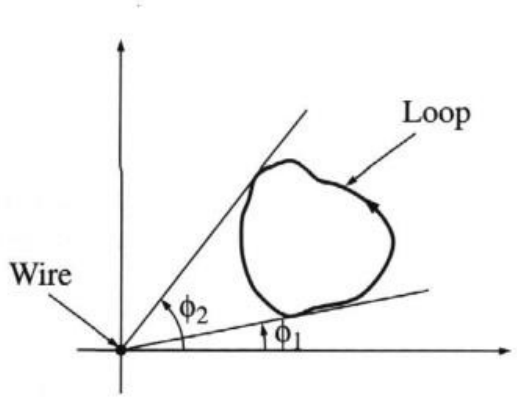
$$\vec{B} = \frac{\mu_0 I}{2\pi R}$$

$$\oint \vec{B} \cdot d\vec{l} = \oint \frac{\mu_0 I}{2\pi R} dl = \frac{\mu_0 I}{2\pi R} \oint dl$$

$$= \mu_0 I$$



$$\vec{B} = \frac{\mu_0 I}{2\pi R}\hat{\varphi} \qquad d\vec{l} = dr\hat{r} + rd\varphi\hat{\varphi} + dz\hat{z}$$



$$\oint \vec{B} \cdot d\vec{l} = \frac{\mu_0 I}{2\pi} \oint \frac{1}{r} r\, d\varphi = \frac{\mu_0 I}{2\pi} \int_0^{2\pi} d\varphi = \mu_0 I$$

$$\longrightarrow \int_{\varphi_1}^{\varphi_2} d\varphi + \int_{\varphi_2}^{\varphi_1} d\varphi = 0$$

$$\therefore \oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}; \quad I_{enc} = \int \vec{J} \cdot d\vec{a}$$

$$\int (\nabla \times \vec{B}) \cdot d\vec{a} \quad \mu_0 \int \vec{J} \cdot d\vec{a} \quad \therefore \nabla \times \vec{B} = \mu_0 \vec{J}$$

5.3.2 The Divergence and Curl of $\vec{B}$

Biot-savart law

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}\times\hat{R}}{R^2}d\tau'$$

$$\nabla\cdot\vec{B} = \frac{\mu_0}{4\pi}\int \nabla\cdot(\vec{J}\times\frac{\hat{R}}{R^2})d\tau',\ d\tau' \to dx'dy'dz'$$

$$\nabla\cdot(J\times\frac{\hat{R}}{R^2}) = \frac{\hat{R}}{R^2}\cdot(\nabla\cdot\vec{J}) - \vec{J}\cdot(\nabla\times\frac{\hat{R}}{R^2})$$

$(\nabla\cdot\vec{J}) = 0$ because $J(x',y',z')$; $(\nabla\times\frac{\hat{R}}{R^2}) = 0$

$$\therefore \nabla\cdot\vec{B} = 0$$

$B(x,y,z)$

$J(x',y',z')$

$$R = (x-x')\hat{i} + (y-y')\hat{j} + (z-z')\hat{k}$$

$$d\tau' = dx'dy'dz'$$

$$\nabla = \hat{i}\partial x + \hat{j}\partial y + \hat{z}\partial z$$

$$\nabla' = \hat{i}\partial x' + \hat{j}\partial y' + \hat{z}\partial z'$$

$$\nabla\times\vec{B} = \frac{\mu_0}{4\pi}\int \nabla\times(\vec{J}\times\frac{\hat{R}}{R^2})d\tau'$$

$$\nabla\times(\vec{A}\times\vec{C}) = (\vec{C}\cdot\nabla)\vec{A} - (\vec{A}\cdot\nabla)\vec{C} + \vec{A}(\nabla\cdot\vec{C}) - \vec{C}(\nabla\cdot\vec{A})$$

$$\nabla\times(\vec{J}\times\frac{\hat{R}}{R^2}) = (\frac{\hat{R}}{R^2}\cdot\nabla)\vec{J} - (\vec{J}\cdot\nabla)\frac{\hat{R}}{R^2} + \vec{J}\underbrace{(\nabla\cdot\frac{\hat{R}}{R^2})}_{4\pi\delta^3(\hat{R})} - \frac{\hat{R}}{R^2}(\nabla\cdot\vec{J})$$

$(\frac{\hat{R}}{R^2}\cdot\nabla)\vec{J} = 0$; $(\vec{J}\cdot\nabla)\frac{\hat{R}}{R^2}$ ($x\leftrightarrow x'$); $(\nabla\cdot\vec{J}) = 0$

5.3.2

$$\frac{\hat{R}}{R^2} = \frac{x-x'}{R^3}\hat{i} + \frac{y-y'}{R^3}\hat{j} + \frac{z-z'}{R^3}\hat{k}$$

$$\because \nabla \cdot (f\vec{A}) = f(\nabla \cdot \vec{A}) + \vec{A} \cdot (\nabla f)$$

$$(\vec{J} \cdot \nabla')(\frac{x-x'}{R^3}) = \nabla' \cdot [\frac{x-x'}{R^3}\vec{J}] - (\frac{x-x'}{R^3})\underbrace{(\nabla' \cdot \vec{J})}_{0}$$ for steady current

$$\int_{volume} \nabla' \cdot [\frac{x-x'}{R^3}\vec{J}]d\tau = \oint_{surface} \frac{x-x'}{R^3}\vec{J} \cdot d\vec{a} \quad \to 0$$

→ To where $\vec{J} = 0$

$$\nabla \times \vec{B} = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}')4\pi\delta^3(\vec{r}-\vec{r}')d\tau' = \mu_0\vec{J}(\vec{r})$$

Ampere's law in differential form

$$\boxed{\nabla \times \vec{B} = \mu_0\vec{J}(\vec{r})}$$

5.3.3 Applications of Ampere's Law

Ampere's Law in differential form

$$\boxed{\nabla \times \vec{B} = \mu_0 \vec{J}}$$

$$\int (\nabla \times \vec{B}) \cdot d\vec{a} = \oint \vec{B} \cdot d\vec{l} = \mu_0 \int \vec{J} \cdot d\vec{a} = \mu_0 I_{enc}$$

Ampere's Law in integral form

$$\boxed{\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}}$$

Electrostatics: Coulomb $\longrightarrow$ Gauss

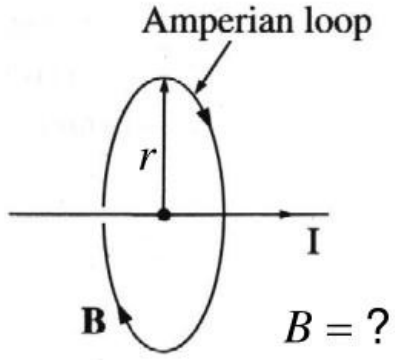
Magnetostatics: Bio-Savart $\longrightarrow$ Ampere

The standard current configurations which can be handled by Ampere's law:

1. Infinite straight lines
2. Infinite planes
3. Infinite solenoids
4. Toroid

5.3.3 (2)

Ex.7

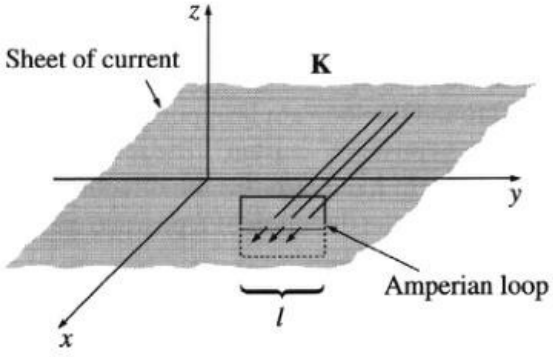


$$\oint_{\parallel \text{symmetry}} \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} = \mu_0 I$$

$$B \oint dl = B \cdot 2\pi r$$

$$B = \frac{\mu_0 I}{2\pi r}$$

Ex.8



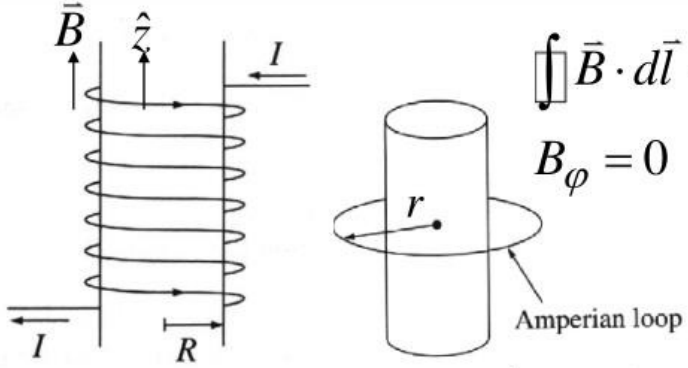
$$B(z)[\hat{j}]$$

$$\oint_{\parallel 2Bl} \vec{B} \cdot dl = \mu_0 I_{enc} = \mu_0 K l$$

$$\vec{B} = \begin{cases} \frac{\mu_0}{2} K\hat{j} & for \quad z < 0 \\ -\frac{\mu_0}{2} K\hat{j} & for \quad z > 0 \end{cases}$$

5.3.3 (3)

Ex.9



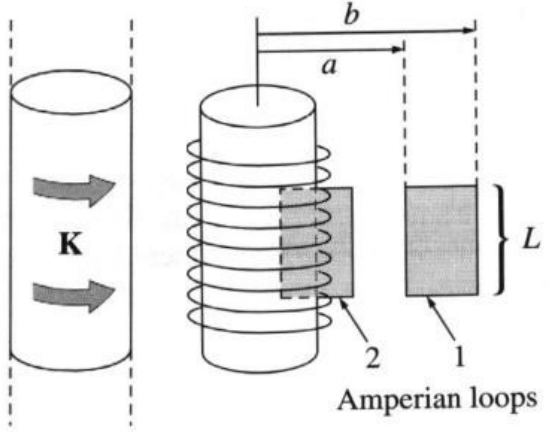
$$\oint \vec{B}\cdot d\vec{l} = B_\varphi(2\pi r) = \mu_0 I_{enc} = 0$$

$$B_\varphi = 0$$

loop 1.

$$\int_1 \vec{B}\cdot d\vec{l} = [B(a) - B(b)]L = \mu_0 I_{enc} = 0$$

$$B(a) = B(b)$$



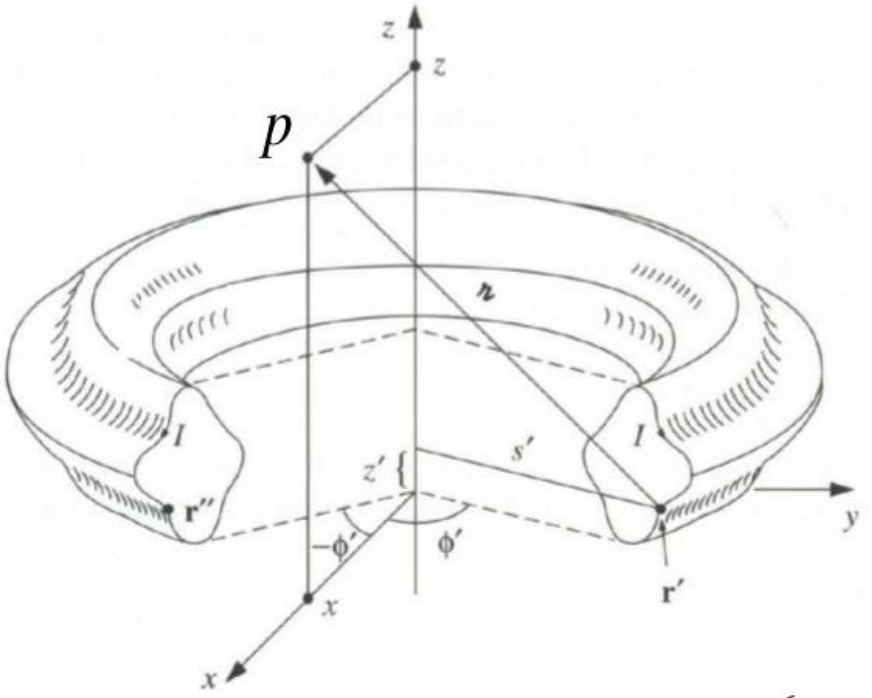
loop 2.

$$\int_2 \vec{B}\cdot d\vec{l} = BL = \mu_0 I_{enc} = \mu_0 NIL$$

$$B\begin{cases} \mu_0 NI\hat{z} & inside \\ 0 & outside \end{cases}$$

5.3.3 (4)

Ex.10



Solution:

$$\oint \vec{B} \cdot d\vec{r} = \mu_0 I_{enc} \qquad B(r) = \begin{cases} \dfrac{\mu_0 nI}{2\pi r}\hat{\varphi} & inside \\ 0 & outside \end{cases}$$

5.3.3 (5)

$\vec{B}(p) = ?$ $\quad d\vec{B} = \frac{\mu_0}{4\pi}\frac{\vec{I}\times\vec{R}}{R^3}dl$ $\quad p = (x_0, 0, z_0)$

$r' = (r\cos\varphi, r\sin\varphi, z)$

$\vec{R} = \vec{p} - \vec{r}\,' = (x_0 - r\cos\varphi, -r\sin\varphi, z_0 - z)$

$\vec{I} = I_r\hat{r} + I_z\hat{z} = (I_r\cos\varphi, I_r\sin\varphi, I_z)$

$(\because I_\varphi = 0)$

$\vec{I}\times\vec{R} = \varepsilon_{ijk} I_i R_j \hat{k}$

$= \hat{x}[I_y R_z - I_z R_y] + \hat{y}[I_z R_x - I_x R_z] + \hat{z}[I_x R_y - I_y R_x]$

$= \hat{x}[I_r\sin\varphi(z_0 - z) - I_z(-r\sin\varphi)] + \hat{y}[I_z(x_0 - r\cos\varphi)$

$-I_r\cos\varphi(z_0 - z)] + \hat{z}[I_r\cos\varphi(-r\sin\varphi) - I_r\sin\varphi(x_0 - r\cos\varphi)]$

$= \hat{x}\{\sin\varphi[I_r(z_0 - z) + I_z r]\} + \hat{y}\{I_z x_0 - \cos\varphi[I_z r + I_r(z_0 - z)]\}$

$+\hat{z}[-\sin\varphi I_r x_0]$

$\hat{x}$ and $\hat{z}$ components cancel out $\because \sin\varphi$ from r' and r''

$\therefore \vec{I}\times\vec{R} = (\quad)\hat{y} = (\quad)\hat{\varphi}$ $\qquad \vec{B}$ *in* $\hat{\varphi}$

5.4.1 The Vector Potential

E.S. : $\nabla \times \vec{E} = 0 \quad \Rightarrow \quad \vec{E} = -\nabla V \qquad V \to V' + const \qquad [\nabla const = 0]$

M.S. : $\nabla \cdot \vec{B} = 0 \quad \Rightarrow \quad \boxed{\vec{B} = \nabla \times \vec{A}} \qquad \vec{A} \to \vec{A}' + ? \qquad [\nabla \times \nabla \lambda = 0]$

a constant-like vector — function

Gauge transformation

$\vec{A}$ is a vector potential in magnetostatics

$$\nabla \times \vec{B} = \nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$$

If there is $\vec{A}'$ that $\nabla \cdot \vec{A}' \neq 0$, can we find a function λ to obtain $\nabla \cdot \vec{A} = 0$ with $\boxed{\vec{A} = \vec{A}' + \nabla \lambda}$?

5.4.1 (2)

$$0 = \nabla \cdot \vec{A} = \nabla \cdot \vec{A}' + \nabla^2 \lambda$$

$$\nabla^2 \lambda = -(\nabla \cdot \vec{A}') \qquad \nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

$$\boxed{\lambda = \frac{1}{4\pi}\int \frac{\nabla \cdot \vec{A}'}{R} d\tau} \qquad V = \frac{1}{4\pi}\int \frac{\rho}{R} d\tau$$

$$\boxed{\nabla^2 \vec{A} = -\mu_0 \vec{J}}$$ Ampere's Law

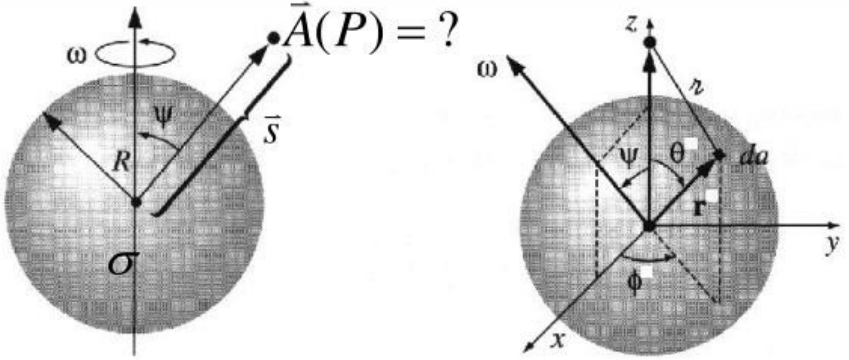
$$\boxed{\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}}{R} d\tau}$$ if $\vec{J}(\infty) = 0$

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}}{R} d\ell = \frac{\mu_0 I}{R}\int \frac{\hat{\ell}}{R} d\ell \; ; \; \vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{K}}{R} da$$

5.4.1 (3)

Example 11

A spiring sphere



Solution:

$$\vec{A}(P) = \frac{\mu_0}{4\pi}\int \frac{\vec{K}}{r} da$$ ← For surface integration over $\frac{1}{r}$ easier

$$\vec{K} = \sigma\,\vec{v}, \quad r = (R^2 + S^2 - 2RS\cos\theta)^{1/2}, \quad da = R^2 \sin\theta\, d\theta\, d\varphi$$

$$(\because \vec{v} \perp \vec{\omega},\ \vec{v} \perp \hat{r}\,)$$

$$\vec{v} = \vec{\omega} \times \vec{r}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega\sin\psi & 0 & \omega\cos\psi \\ R\sin\theta\cos\varphi & R\sin\theta\sin\varphi & R\cos\theta \end{vmatrix}$$

5.4.1 (4)

$$= R\omega[-(\cos\psi\ \sin\theta\ \sin\varphi)\hat{i} + (\cos\psi\ \sin\theta\ \cos\varphi - \sin\psi\ \cos\theta)\,\hat{j} + (\sin\psi\ \sin\theta\ \sin\varphi)\hat{k}]$$

$$\int_0^{2\pi} \sin\varphi\, d\varphi = \int_0^{2\pi} \cos\varphi\, d\varphi = 0$$

$$\therefore \vec{A}(P) = -\left(\frac{\mu_0 R^3 \sigma\omega\sin\psi}{2}\right)\left(\int_0^{\pi} \frac{\cos\theta\sin\theta}{(R^2+S^2-2RS\cos\theta)^{1/2}} d\theta\right)\hat{j}$$

$$(u = \cos\theta)$$

$$\int_{-1}^{+1} \frac{u}{(R^2+S^2-2RSu)^{1/2}} du = \frac{1}{2R^2S^2} \int_{\sqrt{R^2+S^2+2RS}}^{\sqrt{R^2+S^2-2RS}} [y^2 - (R^2+S^2)]dy$$

$$(y = \sqrt{R^2+S^2-2RSu};\ u = \frac{1}{2RS}(R^2+S^2-y^2);\ du = \frac{-ydy}{RS})$$

5.4.1 (5)

$$= \frac{1}{2R^2S^2}\int_{\sqrt{R^2+S^2+2RS}}^{\sqrt{R^2+S^2-2RS}} [y^2 - (R^2+S^2)]dy$$

$$= \frac{y}{2R^2S^2}[\frac{y^2}{3} - (R^2+S^2)]\Bigg|_{R+S}^{|R-S|}$$

$$= \frac{|R-S|}{6R^2S^2}[R^2+S^2-2RS-3(R^2+S^2)] -$$

$$\frac{R+S}{6R^2S^2}[R^2+S^2+2RS-3(R^2+S^2)]$$

$$= -\frac{1}{3R^2S^2}[|R-S|(R^2+S^2+RS) - (R+S)(R^2+S^2-RS)]$$

5.4.1 (6)

if $R > S$

$$= -\frac{1}{3R^2S^2}[(R-S)(R^2+S^2+RS)-(R+S)(R^2+S^2-RS)]$$

$$= -\frac{1}{3R^2S^2}[2RRS-2S(R^2+S^2)]$$

$$= \frac{2S}{3R^2}$$

if $R < S$

$$= -\frac{1}{3R^2S^2}[2SRS-2R(R^2+S^2)]$$

$$= \frac{2R}{3S^2}$$

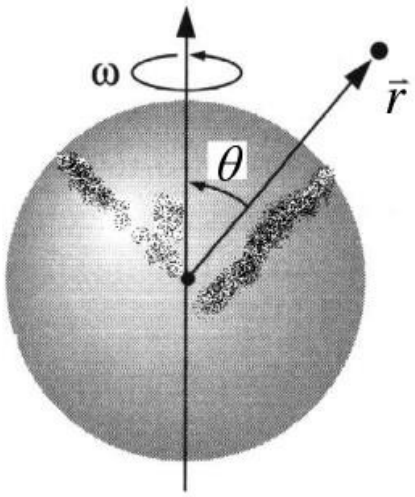
5.4.1 (7)

$$\therefore \vec{A}(P) = \hat{j}\left(-\frac{\mu_0 R^3 \sigma\, \omega \sin\psi}{\cancel{2}}\right) \begin{cases} \dfrac{\cancel{2}S}{3R^2} & R > S \quad ,inside \\ \dfrac{\cancel{2}R}{3S^2} & R < S \quad ,outside \end{cases} \quad for$$

$$\because \vec{\omega} \times \vec{S} = -\omega S \sin\psi\ \hat{j}$$

$$\therefore \vec{A}(P) = \begin{cases} \dfrac{\mu_0 R\sigma}{3}(\vec{\omega}\times\vec{S}) & R > S \quad ,inside \\ \dfrac{\mu_0 R^4\sigma}{3S^3}(\vec{\omega}\times\vec{S}) & R < S \quad ,outside \end{cases} \quad for$$

in



$$\vec{\omega} = \omega\,\hat{z} \qquad \vec{S} = r\hat{r} \qquad \psi \to \theta$$

$$\vec{A}(r,\theta,\varphi) = \begin{cases} \dfrac{\mu_0 R\omega\sigma}{3} r\sin\theta\,\hat{\varphi} & (r \le R) \\ \dfrac{\mu_0 R^4\omega\sigma}{3}\dfrac{\sin\theta}{r^2}\hat{\varphi} & (r \ge R) \end{cases}$$

5.4.1 (8)

Note: $r \le R$

$$\vec{B} = \nabla \times \vec{A} = \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (\sin\theta \, A)\hat{r} - \frac{1}{r} \frac{\partial}{\partial r}(rA)\hat{\theta} \qquad (\vec{A} = A\hat{\varphi})$$

$$= \frac{2\mu_0 R\omega\sigma}{3}(\cos\theta \, \hat{r} - \sin\theta \, \hat{\theta}) = \frac{2}{3}\mu_0 \sigma R \omega \hat{z} = \frac{2}{3}\mu_0 \sigma R \vec{\omega}$$

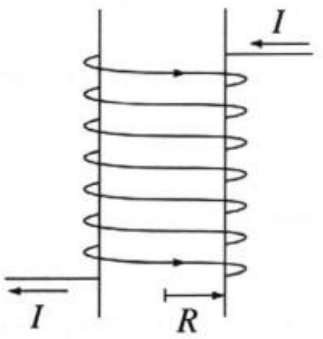
$\vec{B}$ is uniform inside the spherical shell

5.4.1 (9)

Example 12

N turns per unit length

$\vec{A} = ?$



sol :

$$\oint \vec{A}\cdot d\vec{\ell} = \int \nabla\times\vec{A}\cdot d\vec{a} = \int \vec{B}\cdot d\vec{a} = \Phi$$

$$\because \vec{B}\cdot d\vec{\ell} = \mu_0 I_{enc} \Rightarrow \begin{cases} B_{in} = \mu_0 NI \\ B_{out} = 0 \end{cases}$$

$$\oint \vec{A}\cdot d\vec{\ell} = \int \vec{B}\cdot d\vec{a} = \begin{cases} \mu_0 NI(\pi r^2) & r < R \\ \mu_0 NI(\pi R^2) & r > R \end{cases}$$

$\|$

$(\hat{\varphi})A\ 2\pi r$

$$\vec{A} = \begin{cases} \dfrac{\mu_0 NI}{2} r\hat{\varphi} & r < R \\ \dfrac{\mu_0 NI}{2}\dfrac{R^2}{r}\hat{\varphi} & r > R \end{cases}$$

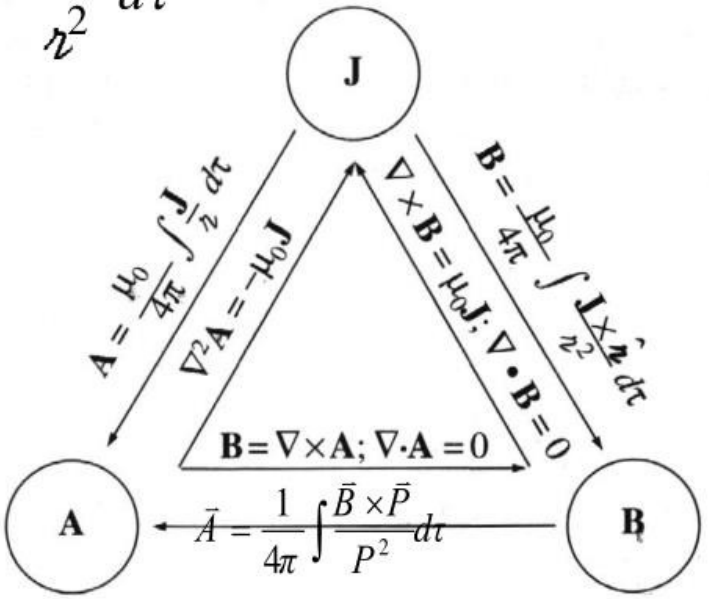
5.4.2 Summary and Magnetostaic Boundary Conditions

$$recall \quad \nabla \cdot \vec{B} = 0 \quad \nabla \times \vec{B} = \mu_0 \vec{J} \quad \Rightarrow \quad \vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{J} \times \hat{R}}{\mathcal{R}^2} d\tau$$

$$\nabla \cdot \vec{A} = 0 \quad \nabla \times \vec{A} = \vec{B}$$

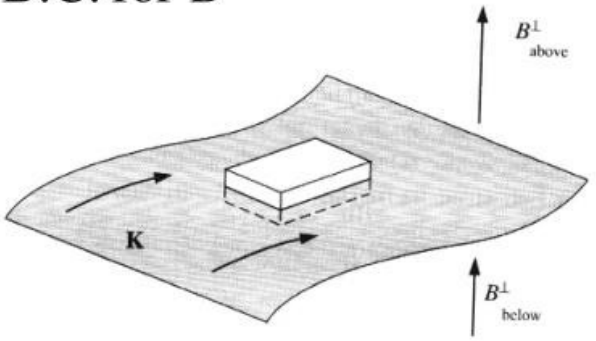
$$\Rightarrow \quad \vec{A} = \frac{1}{4\pi} \int \frac{\vec{B} \times \hat{R}}{\mathcal{R}^2} d\tau$$

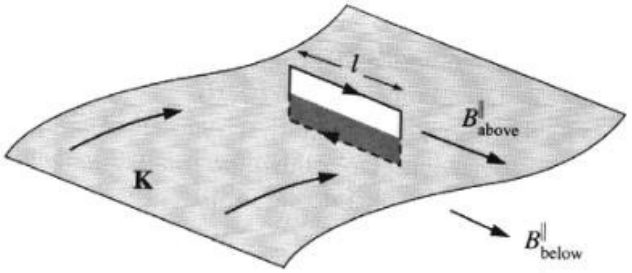
ρ V $\vec{E}$



5.4.2 (2)

B.C. for $\vec{B}$





$$\oint \vec{B} \cdot d\vec{a} = 0$$

$$\oint \vec{B} \cdot d\vec{\ell} = (B^{\parallel}_{above} - B^{\parallel}_{below})\ell$$

$$= \mu_0 I_{enc} = \mu_0 K \ell$$

$$B^{\perp}_{above} = B^{\perp}_{below}$$

$$B^{\parallel}_{above} - B^{\parallel}_{below} = \mu_0 K$$

$$B_{above} - B_{below} = \mu_0 (K \times \hat{n})$$

$$(E^{\parallel}_{a} = E^{\parallel}_{b})$$

$$(E^{\perp}_{a} - E^{\perp}_{b} = \frac{\sigma}{\varepsilon_0}\hat{n})$$

5.4.2 (3)

B.C. for $\vec{A}$

$$\nabla \cdot \vec{A} = 0 \quad \Rightarrow \quad \boxed{A_{above} = A_{below}} \qquad | \qquad (V_a = V_b)$$

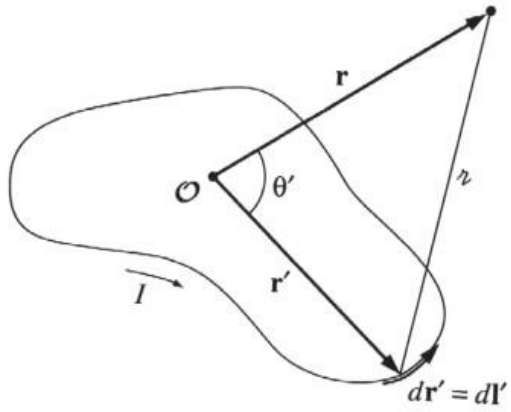
$$\nabla \times \vec{A} = \vec{B}$$

$$(\nabla \times \vec{A})_{above} - (\nabla \times \vec{A})_{below} = \mu_0 (\vec{K} \times \hat{n})$$

$$(\hat{n} \frac{\partial}{\partial n} \times \vec{A})_{above} - (\hat{n} \frac{\partial}{\partial n} \times \vec{A})_{below} = \mu_0 (\vec{K} \times \hat{n})$$

$$\boxed{\frac{\partial \vec{A}_{above}}{\partial n} - \frac{\partial \vec{A}_{below}}{\partial n} = -\mu_0 \vec{K}} \qquad | \qquad \frac{\partial V_a}{\partial n} - \frac{\partial V_b}{\partial n} = -\frac{\sigma}{\varepsilon_0}$$

5.4.3 Multipole Expansion of the Vector Potential



$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}}{R} d\tau' = \underbrace{\frac{\mu_0 I}{4\pi}\oint \frac{1}{R} d\vec{\ell}}_{\text{line current}}$$

$$\frac{1}{R} = \frac{1}{(r^2 + r'^2 - 2rr'\cos\theta)^{1/2}} \overset{r >> r'}{\cong} \frac{1}{r[1-2(r'/r)\cos\theta]^{1/2}} \cong \frac{1}{r}\left(1 + \frac{r'}{r}\cos\theta + \cdots\right)$$

$$\vec{A} \cong \frac{\mu_0 I}{4\pi}\oint \frac{1}{r}\left(1 + \frac{r'}{r}\cos\theta + \cdots\right) d\vec{l} = \frac{\mu_0 I}{4\pi}\left[\underbrace{\frac{1}{r}\oint d\vec{l}}_{=0} + \frac{1}{r^2}\oint r'\cos\theta\, d\vec{l} + \cdots\right]$$

monopole dipole

5.4.3(2)

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0 I}{4\pi r^2}\oint r'\cos\theta d\vec{l} = \frac{\mu_0 I}{4\pi r^2}\oint(\hat{r}\cdot\vec{r}')d\vec{l} \qquad (d\vec{l} = d\vec{r}')$$

$$\hat{r}\times\oint(\vec{r}'\times d\vec{r}') = \oint\vec{r}'(\hat{r}\cdot d\vec{r}') - \oint d\vec{r}'(\hat{r}\cdot\vec{r}')$$

$$= \oint d\left[(\vec{r}'(\hat{r}\cdot\vec{r}')\right] - \oint(\hat{r}\cdot\vec{r}')d\vec{r}' - \oint d\vec{r}'(\hat{r}\cdot\vec{r}')$$

$$= -2\oint(\hat{r}\cdot\vec{r}')d\vec{r}'$$

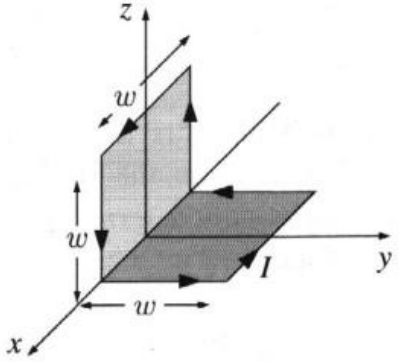
$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0 I}{4\pi r^2}\left[-\frac{1}{2}\hat{r}\times\oint(\vec{r}'\times d\vec{r}')\right]$$

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2} \qquad \vec{m} = \frac{I}{2}\oint(\vec{r}'\times d\vec{l}\,) \qquad \textit{magnetic dipole moment}$$

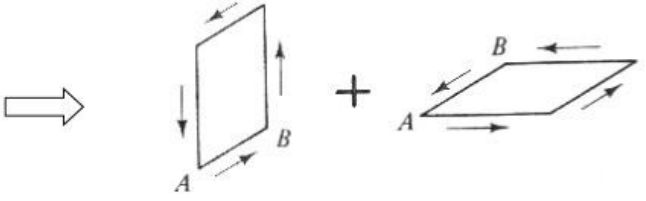
$$\vec{m} = I\vec{a} \qquad \vec{a} = \frac{1}{2}\oint(\vec{r}'\times d\vec{l}\,)$$

5.4.3(3)

Ex. 13



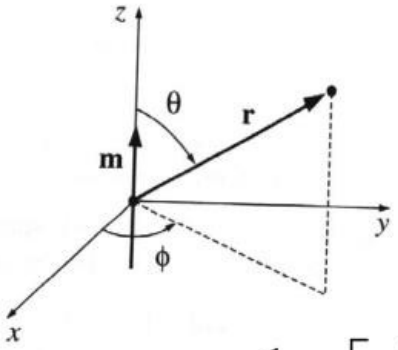
$\vec{m} = ?$



solution:

$$\vec{m} = IS^2\hat{j} + IS^2\hat{k}$$

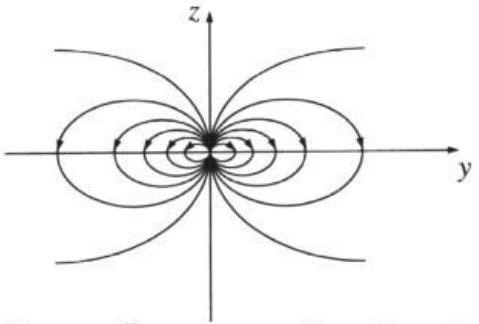
5.4.3(4)



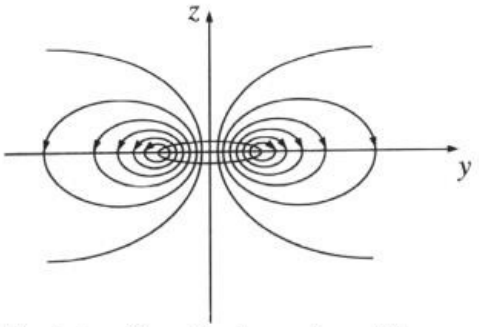
$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2} = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\phi}$$

$$\vec{B} = \nabla\times\vec{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi)\right]\hat{r} + \frac{1}{r}\left[-\frac{\partial}{\partial r}(rA_\phi)\right]\hat{\theta}$$

$$= \frac{\mu_0}{4\pi}\frac{m}{r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta})$$



Field of a “pure” magnetic dipole



Field of a “physical” magnetic dipole