Energy stored in magnetic fields:

The work done on a unit charge against the back emf, in one trip around the circuit is $-\mathcal{E}$ (minus sign for work is done against the back emf). The amount of charge per unit time passing through the wire is I ($\because I = \frac{\text{charge}}{\text{time}}$).

So the work done per unit time is

$$\frac{dW}{dt} = -\mathcal{E} I \qquad \left[\because dW = -\mathcal{E}\, dq \;\Rightarrow\; \frac{dW}{dt} = -\mathcal{E}\frac{dq}{dt} = -\mathcal{E} I \right]$$

$$\Rightarrow \frac{dW}{dt} = LI\frac{dI}{dt} \qquad \left[\because \mathcal{E} = -L\frac{dI}{dt} \right]$$

If we start with zero current and build it up to a value I, the work done is

$$W = \int dW = \int_0^t LI\frac{dI}{dt}\, dt = \int_0^I LI\, dI$$

$$= \frac{1}{2} LI^2$$

Now we know, the flux through a loop is $\Phi = LI$, again

$$\Phi = \int_S \vec{B}.d\vec{a} = \int (\nabla \times \vec{A}).d\vec{a} \qquad \left[\because \vec{B} = \nabla \times \vec{A} \right]$$

$$= \oint_C \vec{A}.d\vec{l}$$

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So, we can write

$$\Phi = LI = \oint_C \vec{A} \cdot d\vec{l}$$

$$\therefore W = \frac{1}{2} LI^2 = \frac{1}{2} I (LI) = \frac{1}{2} I \oint \vec{A} \cdot d\vec{l}$$

$$= \frac{1}{2} \oint_C (\vec{A} \cdot \vec{I}) \, dl$$

The generalization to the volume current is given by

$$W = \frac{1}{2} \int_V (\vec{A} \cdot \vec{J}) \, dV \qquad [\text{volume can't be closed}]$$

But we know,

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

$$\Rightarrow \vec{J} = \frac{1}{\mu_0} (\nabla \times \vec{B})$$

putting the value of $\vec{J}$ in the above equation,

$$W = \frac{1}{2\mu_0} \int_V \{ \vec{A} \cdot (\nabla \times \vec{B}) \} \, dV$$

Now, we know,

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$$

$$\Rightarrow \vec{A} \cdot (\nabla \times \vec{B}) = \vec{B} \cdot \vec{B} - \nabla \cdot (\vec{A} \times \vec{B}) \qquad \left[\because \vec{B} = \nabla \times \vec{A} \right]$$

So, $$W = \frac{1}{2\mu_0} \left[\int_V B^2 \, dV - \int_V \nabla \cdot (\vec{A} \times \vec{B}) \, dV \right]$$

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$$\Rightarrow W = \frac{1}{2\mu_0}\left[\int_V B^2\, dV - \oint_S (\vec{A}\times\vec{B})\cdot d\vec{a}\right]$$

where S is the surface bounding the volume V.

Now if we increase the region (what we are considering) $\vec{A}$ and $\vec{B}$ both decrease (since $\vec{A} \propto \frac{1}{r}$ and $\vec{B} \propto \frac{1}{r^2}$).

To integrate over all space the surface integral goes to ~~Zero~~ Zero. Therefore,

$$W = \frac{1}{2\mu_0}\int_{\text{all space}} B^2\, dV$$

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