Explanations:

T/F1. The series $\sum_{n=1}^{\infty} a_n$ converges whenever $\lim_{n\to\infty} a_n = 0$.

Sol. False.

The Divergence Test guarantees that the series can converge only if the terms tend to 0. However, the converse fails (the harmonic series, for example, is a counter example).

T/F2. The series $\sum_{n=1}^{\infty} a_n$ converges whenever the sequence of partial sums converges to 0.

Sol. True.

By, Definition , a series converges if and only if its sequence of partial sums converge to any number, and zero is acceptable.

T/F3. If $\sum_{n=1}^{\infty} a_n$ converges and $a_n > 0$, then $\sum_{n=1}^{\infty} a_n^2$ converges.

Sol. True.

We have $a_n > 0$ and consequently $a_n < 1$ eventually. Then also $a_n^2 < a_n$ eventually, so that the Comparison Test applies.

T/F4. If $\sum_{n=1}^{\infty} a_n$ converges and $a_n > 0$, then $\sum_{n=1}^{\infty} \sqrt{a_n}$ converges.

Sol. False.

Consider $a_n = \frac{1}{n^2}, \sqrt{a_n} = \frac{1}{n}$. The former converges and the latter diverges.

T/F5. If $\sum_{n=1}^{\infty} a_n$ converges and $a_n > 0$, then $\sum_{n=1}^{\infty} \frac{1+a_n}{1+a_n}$ converges.

Sol. False.

We have, $a_n \to 0$, then $\frac{1+a_n}{1+a_n} \to \frac{1}{2}$. Thus, the latter series terms do not go to 0, hence it diverges.

T/F6. If $\sum_{n=1}^{\infty} a_n$ converges and $a_n > 0$, then $\sum_{n=1}^{\infty} \frac{2^n+a_n}{3^n+a_n}$ converges.

Sol. True.

We have, $a_n \to 0$. Now apply the ratio test.

$$\frac{a_n+1}{a_n} = \frac{2^{n+1}+a_{n+1}}{2^n+a_n} \cdot \frac{3^n+a_n}{3^{n+1}+a_{n+1}}$$

For large n, the a_i are small and this ratio tends then to $\frac{2}{3}$. That is,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{2}{3}$$

The ratio Test gives convergence.

T/F7. $\sum_{n=1}^{\infty} \sin\frac{1}{n^2}$ convergence.

Sol. True.

It is shown in nearly introductory calculus text that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

With that in mind, apply the Limit Comparison Test using the convergent series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ to see that the original series converges also.

T/F8. $\sum_{n=1}^{\infty} \cos \frac{1}{n^2}$ convergence.

Sol. False.

The terms approach $\cos 0 = 1 \neq 0$; apply the Divergence Test.

T/F9. $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$ convergence.

Sol. False.

Note also that $\frac{1}{n^{1+1/n}} \geq \frac{1}{2} \cdot \frac{1}{n}$ and apply the Comparison Test.

T/F10. If p is a fixed real numbers, then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

Sol. True.

(which was proven with the Integral Test).

T/F11. If $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges.

Sol. True.

Apply Abel Test, with a_n as given, $b_n = \frac{1}{n}$.

T/F12. The series $\sum_{n=1}^{\infty} n^{\cos 3}$ converges.

Sol. False.

This is a p-series with $p = -\cos 3 < 1$.

T/F13. $\sum_{n=1}^{\infty} \frac{n+1}{n+5}$ converges to $\frac{1}{5}$.

Sol. False.

The terms approach 1, so the series in fact diverges.

T/F14. $\sum_{n=2}^{\infty} 3(4)^{-n+1}$ converges to 1.

Sol. True.

This is a geometric series with first term $\frac{3}{4}$ and ratio $\frac{1}{4}$. The sum then is

$$\frac{\frac{3}{4}}{1-\frac{1}{4}} = 1$$

T/F15. If $a_n > 0$ and the sequence of partial sums for $\sum_{n=1}^{\infty} a_n$ is bounded above, then the series converges.

Sol. True. A monotone increasing sequence that is bounded above must converge.

T/F16. If the sequence of partial sums for $\sum_{n=1}^{\infty} a_n$ is bounded, then the series converges.

Sol. False.

Consider the series $1 - 1 + 1 - 1 + 1 - 1$ .... Its partial sums oscillate between 1 and 0. Thus, the partial sums are bounded but clearly do not converge.

T/F17. Every rearrangement of an absolutely convergent series has the same sum.

Sol. True.

T/F18. If the Strengthened Root Test indicates absolute convergence for a given series, then the Strengthened Ratio Test will also indicate absolute convergence for that series.

Sol. False.

Consider the series $\sum_{n=1}^{\infty} \frac{1}{k_n^n}$,

where k_n is 2 or 3 according as n is odd or even. The Strengthened Root Test clearly succeeds because the nth roots of terms are given by $\frac{1}{k_n} \leq \frac{1}{2} < 1$. However, the ratio of consecutive terms $\frac{a_{n+1}}{a_n}$ does not yield to analysis because, for n even, we have a ratio of $\frac{3^n}{2^{n+1}} > 1$. In fact the ratios are not even bounded above, so the Strengthened Ratio Test fails.

T/F19. If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Sol. False.

Without assuming the terms are positive this does not follow from the Limit Comparison Test. Take $a_n = \frac{(-1)^n}{\sqrt{n}}, b_n = \frac{1}{n\log n} + \frac{(-1)^n}{n}$. Then $\sum_{n=2}^{\infty} b_n$ diverges because it is the sum of a divergent positive series (by the Abel-Diniscale) and the convergent altering harmonic series, but $\sum_{n=1}^{\infty} a_n$ is a convergent alternating series. Yet

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} \frac{(-1)^n}{\sqrt{n}}}{\frac{1}{n\log n} + \frac{(-1)^n}{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{1 + \frac{(-1)^n}{\log n}} = \infty$$

T/F20. If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Sol. False.

Interchanging the roles of a_n and b_n, this is the contrapositive of the previous item.

T/F21. If $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} |a_n| = B$ and A and B are finite, then $|A| = B$.

Sol. False.

Take $a_n = \left(\frac{1}{2}\right)^n$. $\sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^n = -\frac{1}{3}; \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1$

T/F22. If $\sum_{n=1}^{\infty} a_n$ diverges and $a_n > 0$, then $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ diverges also.

Sol. True.

In the case that $a_n \to 0$, we eventually have

$$\frac{a_n}{1+a_n} > \frac{a_n}{2}$$

and the Comparison Test applies. If on the other hand $a_n \nrightarrow 0$, then the terms $\frac{a_n}{1+a_n}$ do not go to 0 either and the Divergence Test applies.

T/F23. If both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ diverge, then $\sum_{n=1}^{\infty} (a_n + b_n)$ diverges.

Sol. False.

Take $a_n = 1, b_n = -1$.

T/F24. If both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ diverge, then $\sum_{n=1}^{\infty}(a_n + b_n)$ converges.

Sol. False.

Take $a_n = b_n = 1$.

T/F25. If both $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty}(a_n + b_n)$ diverges.

Sol. True.

Let A_n, B_n, C_n be the partial sums of the three series, respectively, and assume to the contrary that $\{C_n\}$ did converge. Then, since $B_n = C_n - A_n$ and the latter two sequences converge, $\{B_n\}$ must converge also, contrary to assumption.

T/F26. If both $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty}(a_n + b_n)$ converges.

Sol. False.

This would contradict the previous item (T/F 25).

T/F27. If $a_n > 0$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} 1/a_n$ diverges.

Sol. True.

Using $\sum_{n=1}^{\infty} a_n$ converges implies that $a_n \to 0$, hence $1/a_n \to \infty$. Then $\sum_{n=1}^{\infty} 1/a_n$ diverges by the Divergence Test.

T/F28. If $a_n > 0$ and $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} 1/a_n$ converges.

Sol. False.

Take $a_n = 1$.

T/F29. If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = 1$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Sol. False.

Again, the Limit Comparison Test does not apply because the terms need not be nonnegative. Take $b_n = \frac{(-1)^n}{\sqrt{n}}$, the terms of a convergent alternating series, and take $a_n = b_n + 1/n$. then $\sum_{n=1}^{\infty} a_n$ is divergent, because it is formed as the sum of a convergent series and a divergent (harmonic) series. We now check

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{(-1)^n}{\sqrt{n}} + \frac{1}{n}}{\frac{(-1)^n}{\sqrt{n}}} = \lim_{n\to\infty} \left(1 + \frac{(-1)^n}{\sqrt{n}}\right) = 1$$

T/F30. If $\lim\limits_{n\to\infty} a_n = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n b_n$ converges.

Sol. False.

Take $a_n = b_n = \frac{(-1)^n}{\sqrt{n}}$.

T/F31. If $\lim\limits_{n\to\infty} a_n = 0, 0 < a_{n+1} \le a_n$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n b_n$ converges.

Sol. True.

Apply Abel's Test.

T/F32. If $\lim_{n\to\infty} a_n = 0$ and $\sum_{n=1}^{\infty} b_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n b_n$ converges.

Sol. True.

Apply the Comparison Test to $\sum_{n=1}^{\infty}|b_n|, \sum_{n=1}^{\infty}|a_n b_n|$, noticing that eventually we must have $|a_n| < 1$.

T/F33. If $a_n \geq 0$, then $\sum_{n=1}^{\infty} a_n$ does not converge conditionally.

Sol. True.

Since $\sum_{n=1}^{\infty}|a_n|, \sum_{n=1}^{\infty} a_n$ are in fact the same series, they cannot have different end behavior.

T/F34. The Alternating Series Test is a test for conditional convergence.

Sol. False.

While the Alternating Series Test is capable of showing that conditionally convergent series are convergent, it does not check that the associated series with all terms positive is divergent. thus it cannot distinguish absolutely alternating series from conditionally convergent alternating series.

T/F35. If $\sum_{n=1}^{\infty} a_n b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge.

Sol. False.

Take $a_n = b_n = \frac{1}{n}$.

T/F36. If $a_n > 0$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \sin a_n$ converges.

Sol. True.

It is shown in nearly every introductory calculus text that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

With that in mind, apply the Limit Comparison Test, noting that $a_n \to 0$.

T/F37. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \cos a_n$ converges.

Sol. False. $a_n \to 0$. Thus $\cos a_n \to \cos 0 = 1 \neq 0$, and the Divergence Test applies.

T/F38. If $0 < a_{n+1} < a_n$ for all n and $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges and has sum S satisfying $|S_n - S| < a_{n+1}$.

Sol. True.

This is the Alternating Series Test .

T/F39. $\sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)^n$ is a divergent series.

Sol. True.

The terms do not tend to 0. (In fact they tend to $1/e$).

T/F40. The Root Test cannot alone be used to determine conditional convergence.

Sol. True.

In order to verify conditional convergence one must show 1) convergence for the series as given, and 2) that the series does not converge absolutely. The Root test may only give information as regards the second requirement.

T/F41. The Ratio Test cannot alone be used to determine conditional convergence.

Sol. True.

In order to verify conditional convergence one must show 1) convergence for the series as given, and 2) that the series does not converge absolutely. The Ratio test may only give information as regards the second requirement.

T/F42. $\sum_{n=1}^{\infty} \frac{1}{n^n} \geq 2$.

Sol. False

We simply observe that $\sum_{n=1}^{\infty} \frac{1}{n^n} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^n} < 1 + \sum_{n=2}^{\infty} \frac{1}{2^n} = \frac{3}{2}$

T/F43. The Ratio Test cannot be used to determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{n}{n^3+5}$.

Sol. True.

The limiting ratio is 1. In fact the same would happen with any series with terms algebraic in n.

T/F44. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges also.

Sol. False.

Consider $a_n = \frac{(-1)^n}{n}$.

T/F45. $\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots + \left(\frac{1}{2}\right)^{10000} < 1$.

Sol. True.

$\sum_{n=1}^{10000} \left(\frac{1}{2}\right)^n < \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1$.

T/F46. If $\sum_{n=1}^{\infty} b_n$ converges and if $0 \leq a_{n+k} \leq b_n$ for some $k \geq 1000$ and every $n \geq 1000$, then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges absolutely.

Sol. True.

Since the behavior of a series is a property of any of its tails, we clearly have $\sum_{n=1}^{\infty} b_{n+k}$ to be convergent since $\sum_{n=1}^{\infty} b_n$ is. The Comparison Test applies.

T/F47. $\sum_{n=1}^{\infty} \frac{1}{\log(n^6+1)}$ converges.

Sol. False.

Apply the Limit Comparison Test against

$$\sum_{n=1}^{\infty} \frac{1}{\log(n^6)} = \sum_{n=1}^{\infty} \frac{1}{6\log(n)}$$

which in turn diverges by comparison with the harmonic series.

T/F48. If $a_n \leq b_n \leq 0$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges also.

Sol. True.

Changing the sign of every term in a series does not affect its convergence. We apply the Comparison Test to the series $\sum_{n=1}^{\infty} -b_n$ and $\sum_{n=1}^{\infty} -a_n$.

T/F49. If $a_n \le b_n \le 0$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges also.

Sol. True.

Changing the sign of every term in a series does not affect its convergence. We apply the Comparison Test to the series $\sum_{n=1}^{\infty} -b_n$ and $\sum_{n=1}^{\infty} -a_n$.

T/F50. If $a_n > 0$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges also.

Sol. True.

By assumption it converges absolutely. Hence it is also convergent.

T/F51. If the condition $0 \le a_{n+1} \le a_n$ fails (even eventually) when applying the Alternating Series Test, then the series $\sum_{n=1}^{\infty} (-1)^n a_n$ diverges.

Sol. False.

Consider $a_n = 1/k_n^n$, where k_n is 2 or 3 according as n is odd or even. By the Strengthened Root Test, $\sum_{n=1}^{\infty} (-1)^n a_n$ converges absolutely. However, $a_{n+1} \le a_n$ never holds for even n.

T/F52. If the condition $\lim_{n\to\infty} a_n = 0$ fails when applying the Alternating Series Test, then the series $\sum_{n=1}^{\infty} (-1)^n a_n$ diverges.

Sol. True.

Use the Divergence Test.

T/F53. If the Ratio Test indicates absolute convergence for a particular series, then the Root Test will also indicates absolute convergence for that series.

Sol. True.

Suppose that the Ratio Test will show convergence for $\sum_{n=1}^{\infty} a_n$, where without loss of generality $a_n \ge 0$. Then consider $r_n = a_n/a_{n-1}$. Since the Ratio Test is decisive, there exists a constant $c < 1$ such that eventually $r_n < c$. Let N be large enough that $r_n < c$ hold for $n \ge N$. Then, for $n \ge N$,

$$a_n^{\frac{1}{n}} = (a_N(a_n/a_N))^{1/n} < (a_N c^{n-N})^{1/n} = c\left(\frac{a_N}{c^N}\right)^{1/n}$$

now the expression in the parentheses is independent of n, so that as n grows the exponential expression will approach 1. Thus, the $a_n^{\frac{1}{n}} < c < 1$ eventually, and the Root Test is decisive.

T/F54. $\sum_{n=1}^{\infty} \frac{n+2}{n+4} = \infty$.

Sol. True.

Since the series is positive, it will suffice to show that the series does not converge. This follows the Divergence Test because the terms do not approach 0.

T/F55. If $\sum_{n=1}^{\infty} a_n = 7$ and $a_1 = a_2 = 3$, then $\sum_{n=3}^{\infty} a_n = 1$.

Sol. True.

Let S_n be the nth partial sum of the first series and let $T_n(T_1 = 0, T_2 = 0)$, be the nth partial sum of the second series. For $n \geq 3, S_n = T_n + 6$. since $S_n \to 7, T_n \to 1$.

T/F56. $\frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \dots, \frac{n+1}{n}, \dots$ is a convergent series.

Sol. False.

It is a convergent sequence. In everyday language sequence and series are used almost interchanging, but not in mathematics. In addition we note that the series $\sum_{n=2}^{\infty} \frac{n+1}{n}$ diverges since $a_n = \frac{n+1}{n} \to 1$.

T/F57. The first three terms in the sequence of partial sums associated with $\sum_{n=1}^{\infty}(1/2)^n$ and $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}$.

Sol. True.

This is a direct computation.

T/F58. $\sum_{n=2}^{\infty}\left(\frac{1}{3n+1} - \frac{1}{3n+4}\right) = \frac{1}{7}$.

Sol. True.

Observe that the series telescopes. The nth partial sum is $\frac{1}{7} - \frac{1}{3n+4}$, which tends to $\frac{1}{7}$.

T/F59. If $\lim_{n\to\infty} a_{2n} = 3$ and $\lim_{n\to\infty} a_{2n+1} = 3$, then $\lim_{n\to\infty} a_n = 3$.

Sol. True.

Fix an $\varepsilon > 0$. Then we can find an N_1, N_2 such that for $n > N_1, |a_{2n} - 3| < \varepsilon$ and for $n > N_2, |a_{2n+1} - 3| < \varepsilon$. Then $|a_n - 3| < \varepsilon$ for all $n > \max\{2N_1, 2N_2 + 1\}$.

T/F60. $\lim_{n\to\infty} a_n = 0 \Leftrightarrow \lim_{n\to\infty} |a_n| = 0$.

Sol. True.

We simply notice that both statements are, by definition, equivalent to the statement that for each $\varepsilon > 0$ we have $|a_n| < \varepsilon$ eventually.

T/F61. If $0 \leq a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges also.

Sol. False.

Take $a_n = \frac{1}{n^2}, b_n = 1$.

T/F62. If a series converges, then no subsequence of the sequence of partial sums can be unbounded.

Sol. True.

Since the series converges, its sequence of partial sums converges. Thus any subsequence would also converge to the same value.

T/F63. The integral test is of no value in testing $\sum_{n=1}^{\infty} \frac{\sin n}{n}$ for convergence or divergence.

Sol. True.

The Integral Test applies only to positive series, and the series in question changes sings erratically.

T/F64. The sum of the infinite series $\sum_{n=1}^{\infty} a_n$ is the limit $S = \lim_{k\to\infty} \sum_{n=1}^{\infty} a_n$, whenever this limit exists.

Sol. True.

T/F65. $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=p}^{\infty} a_n$ converges for every p.

Sol. True.

Each tail of a series will have the same limiting behavior.

T/F66. The Ratio Test is of no value in testing $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)(n+2)}}$ for convergence or divergence.

Sol. True.

Again, a series with algebraic terms cannot be decided with the Ratio Test. We note that the ratios are given by

$$\frac{1/\sqrt{(n+1)(n+2)(n+3)}}{1/\sqrt{n(n+1)(n+2)}} = \frac{\sqrt{n}}{\sqrt{n+3}} \to 1$$

T/F67. $\sum_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right)^{n^2}$ diverges.

Sol. True.

The terms do not tend to 0, so the Divergence Test applies. (In fact they tend to $1/e$).

T/F68. If $a_n > 0$ for all n and if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L < 1$.

Sol. False.

Consider any convergent p-series. (It is also possible for the limit not to exist).

T/F69. If for some constant c we have $ca_n \geq 1/n$, then necessarily $\sum_{n=1}^{\infty} a_n$ diverges.

Sol. True.

Apply the Comparison Test using the harmonic series.

T/F70. The series $\sum_{n=2}^{\infty} (-1)^n$ has as its sequence of partial sums 1,0,1,0, ...

Sol. True.

This is a direct computation.

T/F71. If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} \frac{a_n+2}{a_n^2+1} = 2$.

Sol. False.

The terms approach $2 \neq 0$, so that the series actually diverges by the Divergence Test.

T/F72. If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} \frac{a_n}{a_n^2+n^2}$ converges absolutely.

Sol. True.

Since $a_n \to 0$ we must eventually have $|a_n| < 1$. Then

$$\left|\frac{a_n}{a_n^2+n^2}\right| = \frac{|a_n|}{a_n^2+n^2} \leq \frac{1}{a_n^2+n^2} \leq \frac{1}{n^2}$$

We then apply the Comparison Teat.

T/F73. $\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = 2$.

Sol. True.

We know that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.

Taking a derivative, $\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$ and then multiplying by x, we get

$$\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n$$

The result then follows from taking $x = 1/2$.

T/F74. $\sum_{k=1}^{\infty}\left(Arctan\frac{1}{k} - Arctan\frac{1}{k+1}\right) = \frac{\pi}{4}$.

Sol. True.

Notices that the series telescopes.

T/F75. The Ratio Test is of no help in determining whether $\sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}$ converges or diverges.

Sol. False.

The ratio of consecutive terms is

$$\frac{\frac{(2n+2)!}{((n+1)!)^2}}{\frac{(2n)!}{(n!)^2}} = \frac{(2n+2)(2n+1)}{(n+1)^2} \to 4 > 1.$$

The series diverges by the Ratio Test.

T/F76. If $|a_{n+1}/a_n| > 1$ for all n, then $\sum_{n=1}^{\infty}|a_n|$ diverges.

Sol. True.

We see that the terms $|a_n|$ are actually increasing, and thus are not tending to 0.

T/F77. If $|a_{n+1}/a_n| > 1$ for all n, then $\sum_{n=1}^{\infty}|a_n|$ converges.

Sol. False.

Consider the harmonic series.

T/F78. If $s_n = 3 + 2^{-n}$ then $\lim_{n\to\infty} s_n = 4$.

Sol. False.

The limit is 3.

T/F79. The first three terms in the sequence of partial sums associated to the series $\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n$ are $-\frac{1}{2}, -\frac{1}{4}, -\frac{3}{8}$.

Sol. False.

These are the first three terms of the series. The first three partial sums are $-\frac{1}{2}, -\frac{1}{4}, -\frac{3}{8}$.

T/F80. $\sum_{n=1}^{\infty} 3^{-n}2^{n+1} = 4$.

Sol. True.

This is a geometric series with first term $4/3$ and common ratio $2/3$.

T/F81. Every partial sum of $\sum_{n=1}^{\infty}\left(\frac{1}{n(n+1)}\right)$ is less than 1.

Sol. True.

Writing $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.

we see that the series telescopes, so that the partial sums have the form $1 - \frac{1}{n+1} < 1$.

T/F82. The Integral Test can be used to show that $\sum_{n=3}^{\infty} \frac{(-1)^n}{n \log n \log \log n}$ does not converge absolutely.

Sol. True.

The series $\sum_{n=3}^{\infty} \frac{(-1)^n}{n \log n \log \log n}$ diverges by the Abel-Dini scale, which was proven by the Integral Test.

T/F83. The insertion or removal of a finite number of terms in a series cannot affect its convergence or divergence, although it may affect its sum.

Sol. True.

Regardless of what finite number of terms are changed, some tail sufficiently far out is unaffected, so that the convergence or divergence is preserved.

T/F84. If a series converges, then any summation of a subset of the terms will form a convergent series.

Sol. False.

Let $\sum_{n=1}^{\infty} a_n$ by any divergent positive series whose terms tend to zero. (The harmonic series is an example). Then consider the series $a_1 - a_1 + a_2 - a_2 + a_3 - a_3 + \cdots$ This series converges to 0, but the sum of only the positive terms diverges properly.

As an explicit counterexample, the alternating harmonic series converges but the sub-series formed by the positive terms diverges.

T/F85. If $\sum_{n=1}^{\infty} a_n$ converges and $\{b_n\}_{n=1}^{\infty}$ is a bounded sequence, then $\sum_{n=1}^{\infty} a_n b_n$ converges.

Sol. False.

Let $a_n = b_n = \frac{(-1)^n}{\sqrt{n}}$.

T/F86. If $\sum_{n=1}^{\infty} a_n$ diverges, then If $\sum_{n=1}^{\infty} |a_n|$ must diverges.

Sol. True.

An absolutely convergent series must converge. This states the contra positive.

T/F87. If $a_n = 3/n^2$, then $\sum_{n=1}^{\infty} (a_n - a_{n+1})$ converges to zero.

Sol. False.

The series telescopes, and it is easy to check that the sum is 3.

T/F88. A series representation for the decimal 0.999999 ... can be given by $\sum_{n=1}^{\infty} 9 \left(\frac{1}{10}\right)^n$ where the sum is exactly 1.0, so 0.999999999999 ... = 1.

Sol. True.

The series can be expanded as $0.9 + 0.09 + 0.009 + 0.0009 + \cdots$. As for the sum.

T/F89. If $\lim_{n\to\infty} a_n$ does not exist, then we can always correctly claim that $\sum_{n=1}^{\infty} a_n$ diverges.

Sol. True.

Apply the Divergence Test.

T/F90. A convergent series is either absolutely convergent or conditionally convergent.

Sol. True.

Either $\sum_{n=0}^{\infty}|a_n|$ converges, in which case we have absolute convergence, or else it does not, in which case we have conditional convergence.

T/F91. The alternating harmonic series cannot be rearranged to diverge to $-\infty$.

Sol. False.

T/F92. If $a_n > 0$ and $\sqrt[n]{a_n} < 1$ for all n, then $\sum_{n=1}^{\infty} a_n$ converges.

Sol. False.

Take $a_n = \left(1 - \frac{1}{n}\right)^n$.

T/F93. If $a_n > 0$ and $\sqrt[n]{a_n} \geq 1$ for all n, then $\sum_{n=1}^{\infty} a_n$ diverges.

Sol. True.

We would then have $a_n > 0$, so that the Divergence Test applies.

T/F94. If $a_n > 0$ and $\sqrt[n]{a_n} < r < 1$ for all n and some constant r,then $\sum_{n=1}^{\infty} a_n$ converges.

Sol. True.

This follows from the Root Test.

T/F95. If $\lim_{n\to\infty} |a_{n+1}/a_n| = L < 1$ for some constant L, then $\lim_{n\to\infty} a_n = 0$.

Sol. True.

We first apply the Ratio Test to see that $\sum_{n=1}^{\infty} a_n$ converges.

T/F96. If $\sum_{n=1}^{\infty} a_n^2$ converges, then $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges.

Sol. True.

T/F97. If $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n^p}$ converges for $p > \frac{1}{2}$.

Sol. True.

See $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ converges for $p > \frac{1}{2}$.

T/F98. If $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} \frac{a_n}{1+na_n}$ diverges.

Sol. False.

Consider $a_n = \sqrt{n}$ in the case that $\sqrt{n}$ is an integer and $a_n = 0$ otherwise.

T/F99. If $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} \frac{a_n}{1+na_n}$ converges.

Sol. False.

Consider $a_n = 1$.

T/F100. If $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n^2}$ converges.

Sol. False.

Consider $a_n = n$.

T/F101. If $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n^2}$ diverges.

Sol. False.

Consider $a_n = 2^n$.