(11) (a) $\lim_{x \to 3} \frac{2x+3}{4x-9} = 3$

Let $|x-3| = K \quad x = 3 \pm K \quad u = \left| \frac{2x+3}{4x-9} - 3 \right|$

$$u = \frac{10|x-3|}{|4x-9|}$$

At $x = 3+K \quad u_1 = \frac{10K}{|3+4K|} = \frac{10K}{3+4K}$

At $x = 3-K \quad u_2 = \frac{10K}{|3-4K|} = \frac{10K}{3-4K}$

$\text{Max}\{u_1, u_2\} = \frac{10K}{3-4K}$

$\therefore u < \epsilon$ if $\frac{10K}{3-4K} < \epsilon$ if $10K < 3\epsilon - 12\epsilon K$

if $K(10+12\epsilon) < 3\epsilon$

if $K < \frac{3\epsilon}{10+12\epsilon} = \delta$

$\therefore \delta = \frac{3\epsilon}{10+12\epsilon}$

(b) $\lim_{x \to 6} \frac{x^2-3x}{x+3} = 2$

$$\left| \frac{x^2-3x}{x+3} - 2 \right| = \left| \frac{x^2-5x-6}{x+3} \right|$$

$$= \left| \frac{x^2-6x+x-6}{x+3} \right|$$

$$= \left| \frac{x(x-6)+1(x-6)}{x+3} \right|$$

$$= \frac{|x-6|\,|x+1|}{|x+3|}$$

$|x-6| \nleq \delta_1 = 1$

$5 < x < 7 \quad x+1 < 8 \quad \therefore |x+1| < 8$

$x > 5 \quad x+3 > 8 \quad \therefore |x+3| > 8$

$\therefore \left| \frac{x^2-3x}{x+3} - 2 \right| < \frac{8}{8} |x-6|$

$\therefore \left| \frac{x^2-3x}{x+3} - 2 \right| < \epsilon$ if $|x-6| < \epsilon = \delta_2$

$\delta = \min\{\delta_1, \delta_2\}$

(12) show that following Limits does not exist

(a) $\lim_{x \to 0} \frac{1}{x^2}$ $(x > 0)$

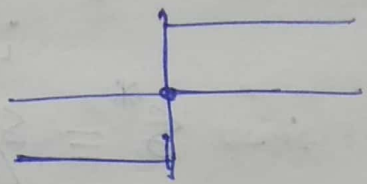
Let $x_n = 1/n \to 0$ But $f(x_n) = n^2 \to \infty$

$\therefore$ Limit does not exist finitely.

(b) $\lim_{x \to 0} \frac{1}{\sqrt{x}}$ $x > 0$ Hints: $x_n = \frac{1}{n^2}$

(c) $\lim_{x \to 0} x + \cancel{sig}\, sgn(x)$

$$sgn(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$



$x_n = \frac{1}{n} \to 0$ $f(x_n) = \frac{1}{n} + \cancel{sig}\, sgn\left(\frac{1}{n}\right)$

$= \frac{1}{n} + 1 \to 1$ as $n \to \infty$

x_n' $\cancel{or}$ $y_n = -1/n \to 0$ as $n \to \infty$

$f(y_n) = -1/n + sgn(-1/n) = -1/n - 1 \to -1$ as $n \to \infty$

$\therefore$ Limit does not exists

(d) $\lim_{x \to 0} \sin\left(\frac{1}{x^2}\right)$ $x > 0$

$x_n = \frac{1}{\sqrt{n\pi}} \to 0$ $f(x_n) = \sin(n\pi) \to 0$

$y_n = \frac{1}{\sqrt{n\pi + \frac{\pi}{2}}} \to 0$ $f(y_n) = \sin(n\pi + \pi/2) \to 1$

$\therefore$ Limit does not exists.

(13) $f: \mathbb{R} \to \mathbb{R}$, $\lim_{x \to 0} f(x) = L$ $g(x) = f(ax)$, $a > 0$

show that $\lim_{x \to 0} g(x) = L$

For Problem 13, let us Consider the following

Let $f(x) = x \quad x \in \mathbb{R} \qquad \lim_{x\to 0} f(x) = 0$

$g(x) = f(ax) = ax \qquad \lim_{x\to 0} g(x) = 0$

For any $\epsilon > 0 \quad \exists\, \delta \equiv \epsilon$

$\therefore |x-0| < \delta \equiv |f(x) - 0| < \epsilon$

$\therefore$ Now, $|g(x) - 0| = |ax - 0| < \epsilon$

if $|x| < \epsilon/a$

$\therefore \delta^* \equiv \delta/a$

For $\epsilon > 0, \ \exists\, \delta > 0 \quad |x-0| < \delta \quad |f(x) - L| < \epsilon$

Now $\delta^* = \delta/a \quad |x-a| < \delta^* \quad |g(x) - L| < \epsilon$.

(13) For any $\epsilon > 0 \quad \exists\, \delta > 0$ s.t

$0 < |x-0| < \delta \Rightarrow |f(x) - L| < \epsilon$

Take: $\delta^* \equiv \delta/a$

$0 < |x-0| < \delta^* \Rightarrow 0 < |ax - 0| < \delta$

$\Rightarrow |f(ax) - L| < \epsilon$

$\therefore \lim_{x\to 0} g(x) = L$.

(14) $\lim_{x\to c} f^2(x) = L$

(a) If $L = 0$ Then $\lim_{x\to c} f(x) = L \equiv 0$

$\forall \epsilon > 0 \quad \exists\, \delta > 0$

$0 < |x-c| < \delta \Rightarrow |f^2(x) - 0| < \epsilon$

$\Rightarrow |f(x)| < \sqrt{\epsilon}$

$\therefore \lim_{x\to c} f(x) = L$ (why?)

(b) $f(x) = \begin{cases} 1 & x \in Q \\ -1 & x \in Q^c \end{cases}$

(15) $f(x) = \begin{cases} x & x \in \mathbb{Q} \cap \mathbb{R} \\ -x & x \in \mathbb{Q}^c \cap \mathbb{R} \end{cases}$

(a) $\lim_{x \to 0} f(x) = 0$

$\forall \epsilon > 0 \quad \exists \delta \equiv \epsilon \quad s.t \quad 0 < |x - 0| < \delta \Rightarrow$

$|f(x) - 0| = |x| < \epsilon$.

(b) $\lim_{x \to c} f(x)$ does not exist if $c \neq 0$

Let $c \in \mathbb{R}$ and $c \neq 0$

$\therefore$ c is a limit point of $\mathbb{Q}$ $\therefore \exists (x_n)$ of rational number s.t $x_n \to c$

$\therefore f(x_n) = x_n \longrightarrow c$ as $n \to \infty$

Again, c is a limit point of $\mathbb{Q}^c$

$\therefore \exists (y_n)$ of $\mathbb{Q}^c$ s.t $y_n \to c$

$\therefore f(y_n) = -y_n \longrightarrow -c$ as $n \to \infty$.

$\because c \neq -c \quad \therefore \lim_{x \to c} f(x)$ does not exist.

(16) $\boxed{\text{Let } f : D \longrightarrow \mathbb{R} \text{ be a function. If } A \subseteq D \text{ then the restriction of } f \text{ to } A \text{ is function } f_A : A \longrightarrow \mathbb{R} \text{ by } f_A(x) = f(x) \ \forall x \in A}$

Let f_1 has limit at $x = c$ $\quad \lim_{x \to c} f_1(x) = L$

$\boxed{\forall \epsilon > 0}$ $\exists \delta > 0 \quad \forall x \in I$ and

$0 < |x - c| < \delta \Rightarrow |f_1(x) - L| < \epsilon$

Now,

$\forall x \in \mathbb{R}$ and $0 < |x - c| < \delta$

$\Rightarrow |f(x) - L| = |f_1(x) - L| < \epsilon$

$\lim_{x \to c} f(x) = L$

Conversely, $\lim_{x \to c} f(x) = L \quad \forall \epsilon > 0 \quad \exists \delta > 0$

$\forall x \in \mathbb{R}$ and $0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon$

$\because c \in I \quad \exists \delta_1 < \delta \quad \therefore N_{\delta_1}(c) \subseteq I$

$\forall x \in N_{\delta_1}(c) \cap I \quad |f_1(x) - L| < \epsilon$

(17) $\lim_{x \to c} f(x) = L$ R.T.P $\lim_{x \to c} f_2(x) = L$

Let (x_n) be any sequence in J s.t

$x_n \to c$

$\therefore$ (x_n) is also a sequence of $\mathbb{R}$

$x_n \to c \Rightarrow f(x_n) \to L$

$\therefore$ Now, $f_2(x_n) = f(x_n) \to L$ as $n \to \infty$

$\therefore$ $\lim_{n \to \infty} f_2(x) = L$

Counter example:

$J = [2021, 2022]$

$$f(x) = \begin{cases} 2022 & x \in J \\ 2021 & x \notin J \end{cases}$$

$\therefore$ $f_2 : [2021, 2022] \longrightarrow \mathbb{R}$ given by

$f_2(x) = f(x) = 2022$

$f_2(x)$ has limit for every $x \in [2021, 2022]$

But $f(x)$ has no limit at $x = 2021$.

H.W: prove 16, by sequential Criterion